Homework 3
Math 336, Winter '00
Due Friday, February 11

From the textbook:

- Page 202: 1, 6
- Page 209: 4abc (give a counterexample if the space is not complete),

Other Problems:

A. Consider the set $\mathcal{M}$ of continuous functions $f : [0,1] \to \mathbf{R}$ with the metric

$$\rho(f,g) = ||f-g||_1 = \int_0^1 |f(x) - g(x)|\, dx$$

is not complete. Give an example of a sequence $\{f_n\}_{n=0}^{\infty} \subset \mathcal{M}$ that converges in this metric but not in the metric $||f-g||_\infty$. (Hint: find a sequence that converges in this metric to a discontinuous function)

B. Two metrics ρ_1, ρ_2 on a set $\mathcal{M}$ are called *comparable* or (in the book's terminology) *uniformly equivalent* if there exist constants $c_1, c_2 > 0$ such that

$$c_1\rho_1(x,y) < \rho_2(x,y) < c_2\rho_1(x,y)$$

for every $x, y \in \mathcal{M}$.

(i): Show that if ρ_1, ρ_2 are comparable metrics on $\mathcal{M}$, then a sequence $\{a_n\}_{n=1}^{\infty}$ converges with respect to the metric ρ_1 if and only if it converges with respect to the metric ρ_2.

(ii): Let $\rho_1, \rho_{max}, \rho_2$ be the metrics described in the textbook for $\mathbf{R}^n$. Show that a sequence converges with respect to one of these metrics if and only if it converges with respect to all of them. (Hint: it's easier to compare the squares of $\rho_1, \rho_2, \rho_{max}$ than it is to compare the metrics themselves.)

(iii): Show that a sequence $\{\mathbf{x}_j\}_{j=1}^{\infty} \subset \mathbf{R}^n$ converges in the Euclidean metric if and only if it converges "coordinate-wise"—i.e. when we write $\mathbf{x}_j = (x_{j,1}, \ldots x_{j,n})$ out in terms of its coordinates, then for each k the sequence $\{x_{j,k}\}_{j=1}^{\infty} \subset \mathbf{R}$ converges.