Solutions for Homework 1

3. Given $\epsilon > 0$, let $\epsilon' = \epsilon/2$. Since $f_n \to f$ uniformly on E, there exists $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies that $|f_n(x) - f(x)| < \epsilon'$ for all $x \in E$. Likewise, there exists $N_2 \in \mathbf{N}$ such that $n \geq N_2$ implies that $|g_n(x) - g(x)| \leq \epsilon'$ for all $x \in E$. Hence, if $N = \max\{N_1, N_2\}$ and $n \geq N$, we have

$$\begin{aligned} |(f_n(x) + g_n(x)) - (f(x) + g(x))| &\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| && (1) \\ &< \epsilon' + \epsilon' = \epsilon. && (2) \end{aligned}$$

This proves that $f_n + g_n$ converges uniformly to $f + g$ on E.

5. Let $[a, b] \subset \mathbf{R}$ be a finite interval, and let $M = \max\{|a|, |b|\}$. I claim that $f_n(x)$ converges uniformly to $f(x) = x^2$ on $[a, b]$. To see this, let $\epsilon > 0$ be given, and choose $N \in \mathbf{N}$ larger than $(2M + 1)/\epsilon$. Then if $n \geq N$ we have

$$\begin{aligned} |f_n(x) - f(x)| &= |(x - 1/n)^2 - x^2| && (3) \\ &= \left| \frac{2x - 1/n}{n} \right| && (4) \\ &\leq \frac{1}{n}(2|x| + 1) && (5) \\ &\leq \frac{2M + 1}{N} < \epsilon. && (6) \end{aligned}$$

for all $x \in [a, b]$. Hence f_n converges uniformly to f on $[a, b]$.

6. Given $\epsilon > 0$, choose $N \in \mathbf{N}$ larger than $1/\epsilon$. Note that by the Mean Value Theorem

$$\frac{\sin(x + 1/n) - \sin(x)}{1/n} = \cos(c)$$

for some c between x and $x + 1/n$. Therefore, if $n \geq N$, we have

$$|\sin(x + 1/n) - \sin(x)| = \frac{|\cos c|}{n} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

Hence $\sin(x + 1/n)$ converges uniformly to $\sin x$ on $\mathbf{R}$.

7. I claim that $f_n(x)$ converges pointwise to 0 on $[0, 1]$. To see this, note that for any given $x \in (0, 1]$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{n} \frac{x}{(x^2 + 1/n^2)} = 0 \cdot \frac{1}{x} = 0.$$

Also, $\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 0 = 0$. This proves my claim for all $x \in [0, 1]$.

To see that f_n does not converge uniformly, I claim that if we take $\epsilon = 1/2$, then no matter how large n is, there will always be some x for which $|f_n(x) - 0| \geq \epsilon$. In fact, if we take $x = 1/n$, then $|f_n(x) - 0| = n/2 \geq 1/2$ for all $n \in \mathbf{N}$. (Incidentally, I found $x = 1/n$ by looking for critical points of f_n. This is actually the point at which f_n achieves its maximum value.)

13. If $-1 < f_n(x) \leq 1$ for all $x \in [0, 1]$, then f_n will converge pointwise to a function $g : [0, 1] \to \mathbf{R}$ such that $g(x) = 0$ if $-1 < f(x) < 1$ and $g(x) = 1$ if $f(x) = 1$.

If $|f_n(x)| < 1$ for all $x \in [0, 1]$, then $f_n(x)$ will converge uniformly to zero. To see that this is true, let $\epsilon > 0$ be given. Since f is continuous on $[0, 1]$, so is $|f(x)|$. Hence, there exists a number $c \in [0, 1]$ such that $|f(x)| \leq |f(c)|$ for all $x \in [0, 1]$. If $f(c) = 0$, there is nothing to prove, since $f_n(x)$ will then be zero for all x and all n. Otherwise, choose $N \in \mathbf{N}$ larger than $\frac{\log \epsilon}{\log |f(c)|}$. If $n \geq N$, we have

$$|f_n(x) - 0| \leq |f(c)|^n < |f(c)|^{\frac{\log \epsilon}{\log |f(c)|}} = \epsilon$$

Hence $f_n(x)$ converges uniformly to zero as claimed.