Solutions for Homework 11

Page 358/ 1b: Solution. Note that $f(u,v) = (0,1)$ if $(u,v) = (0,0)$ (other points are possible). Hence we check,

$$Df_{(0,0)} = \begin{pmatrix} 1 & 1 \\ \cos u & -\sin v \end{pmatrix}\Bigg|_{(u,v)=(0,0)} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

So clearly, f is C^1 near $(0,0)$, and $Df_{(0,0)}$ is invertible. By the inverse function theorem, we conclude that there is a local inverse g defined near $f(0,0) = (0,1)$ and that

$$Dg_{(0,1)} = (Df_{(0,0)})^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

Page 358/ 6: Solution.

a: To check the $1-1$ assertion, suppose that $f(x_1,y_1) = f(x_2,y_2)$ for two points in $(x_1,y_1),(x_2,y_2) \in E$. Then $x_1 + x_2 = y_1 + y_2$ and $x_1 y_1 = x_2 y_2$. Solving the first equation for x_1 gives $x_1 = y_1 + y_2 - x_2$. Substituting this in the second equation and rearranging gives $(y_2 - x_2)(y_2 - y_1) = 0$ So either $y_2 = x_2$ or $y_1 = y_2$. The first case is impossible since $x_2 > y_2$ (by definition of E). The second case implies that $x_1 = x_2$, as well.

To check the onto assertion, let $(s,t) \in^2$ be a point satisfying $s > 2\sqrt{t}$ and $t > 0$. We find $(x,y) \in E$ such that $f(x,y) = (s,t)$ by solving

$$\begin{aligned} x + y &= s \\ xy &= t \end{aligned}$$

for x and y. This can be done by solving the first equation for x, substituting the result in the second equation and solving for y. The result is

$$\begin{aligned} x &= \frac{s + \sqrt{s^2 - 4t}}{2} \\ y &= \frac{s - \sqrt{s^2 - 4t}}{2}, \end{aligned}$$

where our choice of signs is governed by the requirement that $x > y$. We note that since $s \geq 2\sqrt{t}$, then $s, s^2 - 4t > 0$. Since $t > 0$, we have $s^2 - 4t < s^2$. Hence $x > y > 0$ as desired. Thus

$$f^{-1}(s,t) = (\frac{s + \sqrt{s^2 - 4t}}{2}, \frac{s - \sqrt{s^2 - 4t}}{2}).$$

b:

$$Df^{-1}_{f(x,y)} = (Df_{(x,y)})^{-1} = \begin{pmatrix} 1 & 1 \\ y & x \end{pmatrix}^{-1} = \frac{1}{x-y}\begin{pmatrix} x & -1 \\ -y & 1 \end{pmatrix}.$$

c: Using the formula from part (a) gives

$$Df^{-1}_{(s,t)} = \begin{pmatrix} \frac{1}{2} + \frac{s}{2\sqrt{s^2-4t}} & \frac{-1}{\sqrt{s^2-4t}} \\ \frac{1}{2} - \frac{s}{2\sqrt{s^2-4t}} & \frac{1}{\sqrt{s^2-4t}} \end{pmatrix}.$$

Now we substitute $(s,t) = (x+y, xy)$ and obtain

$$Df^{-1}_{f(x,y)} = \begin{pmatrix} \frac{1}{2} + \frac{x+y}{2\sqrt{(x+y)^2-4xy}} & \frac{-1}{\sqrt{(x+y)^2-4xy}} \\ \frac{1}{2} - \frac{x+y}{2\sqrt{(x+y)^2-4xy}} & \frac{2}{\sqrt{(x+y)^2-4xy}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{x+y}{2\sqrt{(x-y)^2}} & \frac{-1}{\sqrt{(x-y)^2}} \\ \frac{1}{2} - \frac{x+y}{2\sqrt{(x-y)^2}} & \frac{1}{\sqrt{(x-y)^2}} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{x}{x-y} & \frac{-1}{x-y} \\ \frac{-y}{x-y} & \frac{1}{x-y} \end{pmatrix}$$

which agrees with part (b).

Other Problems

A. In problem 1b, use the method described in class to approximate a solution (x,y) of $f(x,y) = (.1, 1.2)$. Then use your first approximation to generate a second approximation.

Solution: Taking $\mathbf{x}^0 = (0,0)$ and $\mathbf{y} = (.1, 1.2)$, we use the formula

$$\mathbf{x}^1 = \mathbf{x}^0 + (Df_{\mathbf{x}^0})^{-1} \cdot (\mathbf{y} - f(\mathbf{x}^0)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \cdot \left(\begin{pmatrix} .1 \\ 1.2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} .2 \\ -.1 \end{pmatrix}$$

to generate a new guess. Applying the same formula one more time gives

$$\mathbf{x}^2 = \begin{pmatrix} .2 \\ -.1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ \cos .2 & \sin(-.1) \end{pmatrix}^{-1} \cdot \left(\begin{pmatrix} .1 \\ 1.2 \end{pmatrix} - \begin{pmatrix} .1 \\ 1.19367 \end{pmatrix} \right) = \begin{pmatrix} .207187 \\ -.107187 \end{pmatrix}$$

B. This problem is designed to show what can go wrong when the hypotheses of the inverse function theorem fail.

(1) Define $f :^2 \to^2$ by $f(x,y) = (x^2+y^2, x-2y)$.

(2) At which points $(x,y) \in^2$ are the hypotheses of the inverse function theorem satisfied?

Solution: Since $\det Df_{(x,y)} = -4x - 2y$, the hypotheses of the inverse function theorem are satisfied (i.e Df is invertible) if and only if $y \neq -2x$.

(3) Pick any point (your choice—I want you to use actual numbers here) $(x_0, y_0) \in^2$ where the hypthoses of the inverse function theorem fail, and let $(r_0, s_0) = f(x_0, y_0)$. Pick a number r slightly larger than r_0. How many real solutions (x,y) of $f(x,y) = (r, s_0)$ are there near (x_0, y_0)? What if r is slightly less than r_0?

Solution (example): Take $(x_0, y_0) = (1, -2)$. Then $f(x_0, y_0) = (5, 5)$. Set $r = 5.1$. Then $f(x,y) = (5.1, 5)$ implies that $x = 2y + 5$ and $(2y+5)^2 + y^2 = x^2 + y^2 = 5.1$. Simplifying this last equation and solving for y using the quadratic formula gives

$$y = \frac{-20 \pm \sqrt{400 - 20 \cdot 19.9}}{10} = -2 \pm .141421$$

Hence, we obtain *two* nearby points

$$(x,y) = (1.28284, -1.85858) \qquad \text{and} \qquad (x,y) = (0.717157, -2.14142)$$

satisfying $f(x,y) = (5.1, 5)$. If, on the other hand, we set $r = 4.9$, we obtain

$$y = \frac{-20 \pm \sqrt{400 - 20 \cdot 20.1}}{10} = -2 \pm \frac{sqrt{-2}}{10},$$

so there are *no* points (x,y) satisfyin $f(x,y) = (4.9, 5)$.

(4) Explain why your answers to the last part rule out the existence of a local inverse for f. Explain what's going on here in geometric terms.

Solution: If there were a local inverse g for f defined near $(5, 5)$, we would have that $f(g(\mathbf{y})) = \mathbf{y}$ for every $\mathbf{y}$ near $(5, 5)$—in particular, $(x, y) = g(4.9, 5)$ would satisfy $f(x, y) = (4.9, 5)$. Since there is no such point, g cannot exist.

In geometric terms, solving $f(x, y) = (a, b)$ amounts to finding the intersection of a circle of radius $\sqrt{a}$ and center $(0, 0)$ with the line $x - 2y = b$. The hypotheses of the inverse function theorem fail precisely when (x_0, y_0) is chosen so that the circle and the line defined by equating first and second coordinates of $f(x_0, y_0) = f(x, y)$ are tangent to each other. When the circle is expanded or contracted slightly (i.e. r is chosen slightly larger or slightly smaller), the single intersection either splits into two intersections or disappears altogether.