Solutions for Homework 2

Page 173/ #2:

If we do the integral first, we get

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 \left(1+\frac{x}{n}\right)^n dx &= \lim_{n\to\infty}\frac{n}{n+1}\left(1+\frac{x}{n}\right)^{n+1}\Bigg]_0^1 \\ &= \lim_{n\to\infty}\frac{n}{n+1}\left(\left(1+\frac{1}{n}\right)^{n+1}-1\right) \\ &= 1\cdot(e-1)=e-1.\end{aligned}$$

If we take the limit first, we get

$$\lim_{n\to\infty}\int_0^1 \left(1+\frac{x}{n}\right)^n dx = \int_0^1 e^x\,dx = e-1.$$

Page 174/ #5:

First I will show that if I set $f_n(x) = (f(x))^n$, then $f_n(x) \to 0$ uniformly on $[a,b]$. Since f is continuous, $|f|$ is also continuous. Since $[a,b]$ is closed, we know that there exists a point $c \in [a,b]$ such that $|f(x)| \le |f(c)|$ for every x in $[a,b]$. Therefore, if $\epsilon > 0$ is given, I choose $N \in \mathbf{N}$ larger than $\log\epsilon/\log|f(c)|$. Then $n \ge N$ implies that

$$|f_n(x)-0| = |f(x)|^n \le |f(c)|^n \le |f(c)|^N \le |f(c)|^{\frac{\log\epsilon}{\log|f(c)|}} = \epsilon$$

for every $x \in [a,b]$. Note that the last two inequalities hold because of the hypotheses that $|f(c)| < 1$. This proves that $f_n(x)$ converges uniformly to zero on $[a,b]$.

Now I can invoke theorem 5.2.2 to obtain

$$\lim_{n\to\infty}\int_a^b (f(x))^n\,dx = \lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b 0\,dx = 0,$$

as desired. □

Page 174/ #6:

Let $f : [0,1] \to \mathbf{R}$ be the limit of the sequence $\{f_n\}$. By Theorem 5.2.1, we know that f is continuous. Since continuous functions on closed intervals are bounded, there exists a number $M \in \mathbf{R}$ such that $|f(x)| \le M$ for all $x \in [0,1]$. Now let ϵ be a fixed positive number. For the sake of this argument, we can take $\epsilon = 1$. Then by the definition of uniform converge, there exists a number $N \in \mathbf{N}$ such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n \ge N$ and all $x \in [0,1]$. Therefore

$$|f_n(x)| = |f_n(x) - f(x) + f(x)| \le |f_n(x) - f(x)| + |f(x)| < 1 + M$$

for all $n \ge N$ and all $x \in [0,1]$. On the other hand, for each $n < N$, continuity of f_n implies that there exists a constant $M_n \in \mathbf{R}$ such that $|f_n(x)| \le M_n$ for all $x \in [0,1]$. If we let $K = \max\{1+M, M_1, M_2, \ldots, M_{n-1}\}$, then it follows from the above arguments that $|f_n(x)| \le K$ for all $n \in \mathbf{N}$ and all $x \in [0,1]$. □

Page 181/ #4:

(A) $||f||_1 = \int_a^b |f(x)|\,dx \geq \int_a^b 0\,dx = 0$ for all continuous functions $f : [a,b] \to \mathbf{R}$. Clearly $||f||_1 = 0$ if $f(x) = 0$ for all $x \in [a,b]$. If on the other hand $||f||_1 = 0$, then since $|f|$ is continuous and non-negative, we can apply problem 3b on page 94 to conclude that $f(x) = 0$ for every $x \in [a,b]$. In summary, $||f||_1 \geq 0$ with equality if and only if $f(x) = 0$ for all $x \in [a,b]$.

(B) If $\alpha \in \mathbf{R}$ is given, then

$$||\alpha f||_1 = \int_a^b |\alpha f(x)|\,dx = |\alpha| \int_a^b |f(x)|\,dx = |\alpha| \cdot ||f||_1.$$

(C) If f and g are two continuous functions defined on $[a,b]$, then

$$||f+g||_1 = \int_a^b |f(x)+g(x)|\,dx \leq \int_a^b |f(x)| + |g(x)|\,dx = ||f||_1 + ||g||_1.$$

□

Page 181/ #5

(A) Let $x \in [0,1]$ and $\epsilon > 0$ be given. If $x \leq 1/2$, choose any $N \in \mathbf{N}$. If $n \geq N$, then

$$|f_n(x) - f(x)| = |1-1| = 0 < \epsilon$$

If on the other hand $x > 1/2$, choose $N \in \mathbf{N}$ larger than $(x - 1/2)^{-1}$. Then $n \geq N$ implies that $x > 1/2 + 1/n$. In particular,

$$|f_n(x) - f(x)| = |0-0| = 0 < \epsilon.$$

For all $x \in [0,1]$, therefore, we see that $\lim_{n\to\infty} f_n(x) = f(x)$. In other words f_n converges pointwise to f on $[0,1]$. □

(B) Since both $f_n(x)$ and $f(x)$ are always between 0 and 1, it's clear that $||f - f_n||_\infty \leq 1$. On the other hand, for any given n, we have for $1/2 < x < 1/2 + 1/n$ that $|f_n(x) - f(x)| = 1 - n(x - 1/2)$. Hence

$$||f - f_n||_\infty \geq |f(x) - f_n(x)| = 1 - n(x - 1/2).$$

By letting x tend toward 1/2, we obtain that $||f - f_n||_\infty \geq 1$. Therefore the only possibility is that $||f - f_n||_\infty = 1$ for each $n \in \mathbf{N}$. In particular $||f - f_n||$ does not converge to zero as $n \to \infty$, so the convergence of f_n to f is not uniform. □

(C) Using Theorem 5.2.1 we could have predicted that the $\lim_{n\to\infty} ||f - f_n||_\infty \neq 0$, because if the limit were zero, we would know that f_n converges uniformly to f on $[0,1]$. The theorem would then imply that f is a continuous function. But f has a discontinuity at 1/2, so the convergence can't be uniform.

(D) This is best accomplished by direct computation.

$$\begin{aligned}
\lim_{n\to\infty} ||f_n - f||_1 &= \lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|\, dx \\
&= \lim_{n\to\infty} \int_{1/2}^{1/2+1/n} |1 - n(x - 1/2) - 0|\, dx \\
&= \lim_{n\to\infty} \int_{1/2}^{1/2+1/n} 1 - n(x - 1/2)\, dx \\
&= \lim_{n\to\infty} \frac{1}{2n} = 0.
\end{aligned}$$

□