Solutions for Homework 3

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(i) Consider points $(x_1, y_1), (x_2, y_2) \in \mathbf{R}^2$. Then for the metric ρ_{max}, we have

$$\rho_{max}((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\} \geq 0,$$

since we are taking the maximum of two non-negative numbers. Moreover, the maximum is zero if and only if both absolute values vanish, in which case $x_1 = x_2$ and $y_1 = y_2$. For the metric ρ_1, we have

$$\rho_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2| \geq 0$$

since we are adding non-negative numbers. Moreover, the sum is zero if and only if both absolute values are zero, in which case $x_1 = x_2$ and $y_1 = y_2$ again.

(ii) For the metric ρ_{max} we have

$$\begin{aligned}\rho_{max}((x_1, y_1), (x_2, y_2)) &= \max\{|x_1 - x_2|, |y_1 - y_2|\} \\ &= \max\{|x_2 - x_1|, |y_2 - y_1|\} \\ &= \rho_{max}((x_2, y_2), (x_1, y_1)).\end{aligned}$$

The argument for the metric ρ_1 is similar.

(iii) Let $(x_3, y_3) \in \mathbf{R}^2$ be a third point. Then

$$\begin{aligned}\rho_{max}((x_1, y_1), (x_3, y_3)) &= \max\{|x_1 - x_3|, |y_1 - y_3|\} \\ &\leq \max\{|x_1 - x_2| + |x_2 - x_3|, |y_1 - y_2| + |y_2 - y_3|\} \\ &\leq \max\{|x_1 - x_2|, |y_1 - y_2|\} + \max\{|x_2 - x_3|, |y_2 - y_3|\} \\ &= \rho_{max}((x_1, y_1), (x_2, y_2)) + \rho_{max}((x_2, y_2), (x_3, y_3))\end{aligned}$$

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The facts that $\rho(x, y) = \rho(y, x)$ and that $\rho(x, y) \geq 0$ are pretty clear. Moreover, we have $\rho(x, y) = 0$ if and only if

$$\left|\frac{y - x}{xy}\right| = 0$$

which only happens if $x = y = 0$ (remember that we're assuming $x, y \neq 0$. Finally,

$$\begin{aligned}\rho(x, z) = \left|\frac{1}{x} - \frac{1}{y}\right| & \\ &= \left|\frac{1}{x} - \frac{1}{y} + \frac{1}{y} - \frac{1}{z}\right| \\ &\leq \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} - \frac{1}{z}\right| \\ &= \rho(x, y) + \rho(y, z),\end{aligned}$$

for all $x, y, z \in (0, \infty)$, so the triangle inequality holds, too. □

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The sets in (b) and (c) are complete, but the one in (a) is not. For instance, the sequence $\{(1-1/n,0)\}_{n=1}^{\infty}$ converges to the point $(1,0) \in \mathbf{R}^2$. Therefore it is Cauchy. But $(1,0)$ does not belong to the original set, so the set is not complete.

Other Problem A.

Many answers to this question are possible. Here's mine. Let $f_n : [0,1] \to \mathbf{R}$ be given by $f_n(x) = x^n$. Then $f_n(x)$ converges pointwise to the function $f : [0,1] \to \mathbf{R}$ satisfying $f(1) = 1$ and $f(x) = 0$ for $x \in [0,1)$. If f_n converged uniformly, then it would converge pointwise to the same limiting function, and since f_n is continuous for every n, the limiting function would be continuous. But the pointwise limit f is not continuous, so we conclude that f_n does not converge uniformly. On the other hand, if $\rho(f,g) = \int_0^1 |f(x) - g(x)|\, dx$, then

$$\rho(f_n, f) = \int_0^1 |x^n - 0| = \frac{1}{n+1}$$

for every n (note that the fact that $f(1) = 1$ won't affect the integral). So $\lim_{n\to\infty} \rho(f_n, f) = 0$. Therefore f_n converges to f with respect to the metric ρ.

Other Problem B.

(i) Suppose that $\{a_n\}_{n=1}^{\infty}$ converges uniformly to an element $a \in \mathcal{M}$ with respect to the metric ρ_1. Then given $\epsilon > 0$, I set $\epsilon' = \epsilon/c_2$ and choose $N \in \mathbf{N}$ large enough that $n \geq N$ implies that

$$\rho_1(a_n, a) < \epsilon'.$$

Then for $n \geq N$, we also have

$$\rho_2(a_n, a) \leq c_2 \rho_1(a_n, a) < c_2 \epsilon' = \epsilon.$$

Hence, $a_n \to a$ with respect to the metric ρ_2. Repeating the argument with ρ_1 and ρ_2 switched and with c_2 replaced by $1/c_1$ takes care of the other direction. □

(ii) Let $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbf{R}^n$ be points. Let j between 1 and n be chosen so that $|x_j - y_j|$ is as large as possible. Then we have

$$\begin{aligned}
\rho_{max}(\mathbf{x},\mathbf{y})^2 &= |x_j - y_j|^2 \\
&\leq \sum_{k=1}^{n} |x_k - y_k|^2 = \rho_2(\mathbf{x},\mathbf{y})^2. \\
&\leq \sum_{k,\ell=1}^{n} |x_k - y_k||x_\ell - y_\ell| \\
&= \left(\sum_{k=1}^{n} |x_k - y_k|\right)^2 = \rho_1(\mathbf{x},\mathbf{y})^2 \\
&\leq (n|x_j - y_j|)^2 = n^2 \rho_{max}(\mathbf{x},\mathbf{y})^2.
\end{aligned}$$

Taking square roots (all distances are non-negative so absolute value signs are unnecessary) and summarizing gives:

$$\rho_{max}(\mathbf{x},\mathbf{y}) \leq \rho_2(\mathbf{x},\mathbf{y}) \leq \rho_1(\mathbf{x},\mathbf{y}) \leq n\rho_{max}(\mathbf{x},\mathbf{y}).$$

So ρ_1 and ρ_2 are each comparable to ρ_{max} (and hence comparable to each other as well). Therefore by part (i) a sequence converges with respect to one of these three metrics if and only if it converges with respect to the other two. $\square$

(iii) By part (ii) of this problem, we know that the Euclidean metric ρ_2 is comparable to the box metric ρ_{max}. So if $\mathbf{x}_j \to \mathbf{y} = (y_1, \ldots, y_n) \in \mathbf{R}^n$, we then have for each $1 \leq k \leq n$ and all $j \in \mathbf{N}$ that

$$0 \leq |x_{j,k} - y_k| \leq \rho_{max}(\mathbf{x}_j, \mathbf{y}) \leq c_1 \rho_2(\mathbf{x}_j, \mathbf{y}),$$

for some $c_1 > 0$. Since the right side tends to zero as $j \to \infty$, the Squeeze Theorem tells us that $\lim_{j\to\infty} |x_{j,k} - y_k| = 0$—in other words, for each k between 1 and n, we have $\lim_{j\to\infty} x_{j,k} = y_k$.

Now suppose that $\mathbf{x}_n \to \mathbf{y}$ coordinate-wise. Then we can use comparability of the metrics ρ_1 and ρ_2 to obtain

$$0 \leq \rho_2(\mathbf{x}_j, \mathbf{y}) \leq c_2 \rho_1(\mathbf{x}_j, \mathbf{y}) = c_2(|x_{j,1} - y_1| + \cdots + |x_{j,n} - y_n|).$$

By assumption, all n terms on the right side go to zero as $j \to \infty$, so the Squeeze Theorem allows us to conclude that $\rho_2(\mathbf{x_j}, \mathbf{y})$ goes to zero as well. In other words, $\mathbf{x_j} \to \mathbf{y}$ in the Euclidean metric. $\square$