Solutions for Homework 4

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$K = ||\partial f/\partial y||_\infty = ||\cos y||_\infty = 1$. Also, $M = ||f||_\infty = 1$. The proof shows that we obtain a solution on $[t_0 - \delta, t_0 + \delta]$ so long as both $K\delta$ and $M\delta$ are less than 1. In particular, we will be OK if we take $\delta = 1/2$.

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We just need to find those points $(0, y_0)$ at which the functions f and $\partial f/\partial y$ are continuous. So the answers are

a: y_0 can be anything;
b: $y_0 > 2$;
c: $y_0 \neq 0$;
d: $|y_0| < 1$;
e: $y \neq 0$.

Other Problem A:

Part 1. Given $x, y \in \mathbf{R}$, the Mean Value Theorem guarantees a point c between x and y such that

$$\frac{T(x) - T(y)}{x - y} = T'(c) = \frac{c}{\sqrt{1 + c^2}}.$$

Hence

$$|T(x) - T(y)| = \frac{|c|}{\sqrt{c^2 + 1}}|x - y| < |x - y|.$$

□

Part 2. A fixed point x for T satisfies the equation $\sqrt{x^2 + 1} = x$. Squaring both sides and subtracting x^2 gives. $1 = 0$, which is impossible. Therefore, T has no fixed points. □

Part 3. Note that $T(x) = \sqrt{1 + x^2} > \sqrt{x^2} = |x| \geq x$ for all $x \in \mathbf{R}$. Hence, given $x_1 \in \mathbf{R}$, the sequence $x_1, x_2 = T(x_1), x_3 = T(x_2), \ldots$ is increasing. Either it diverges to infinity as n increases or it converges to its least upper bound. Suppose (by way of contradiction) that $M = \lim_{n\to\infty} x_n = \sup\{x_n\}_{n=0}^{\infty} < \infty$ is the least upper bound. Then since T is a continuous function, we have $\lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = T(M)$. On the other hand, $T(x_n) = x_{n+1} \leq M$ for every n since M is an upper bound for the x_n's. Therefore, $T(M) = \lim_{n\to\infty} T(x_n) \leq M$. But we already know that $T(M) > M$ for any number M, so M must not exist. We conclude that the sequence $\{x_n\}$ diverges to infinity. □

Other Problem B:

Part 1. If $y_1, y_2 : [a, b] \to \mathbf{R}$ are continuous functions, then for each $t \in [a, b]$, we have

$$\begin{aligned} |Ty_1(t) - Ty_2(t)| &= \left| \int_a^t p(s)(y_1(s) - y_2(s)\,ds \right| \\ &\le \int_a^t |p(s)||y_1(s) - y_2(s)|\,ds \\ &\le ||y_1(s) - y_2(s)||_\infty \int_a^t |p(s)|\,ds \\ &\le ||y_1(s) - y_2(s)||_\infty \int_a^b |p(s)|\,ds. \end{aligned}$$

That is,

$$||Ty_1 - Ty_2||_\infty \le \left(\int_a^b |p(s)|\,ds \right) ||y_1 - y_2||_\infty,$$

and T is a contraction provided that the integral on the right side is less than one. □

Part 2. We can choose $b > a$ to be any number such that

$$1 > \int_a^b |p(s)|\,ds = \int_0^b s\,ds = b^2/2.$$

So we can take b to be any number smaller than $\sqrt{2}$. If $y_1(t) = 1$, then

$$\begin{aligned} y_2(t) &= \int_0^t (s \cdot 1 + e^s)\,ds + 1 = \frac{t^2}{2} + e^t \\ y_3(t) &= \int_0^t \left(\left(\frac{s^2}{2} + e^s \right) \cdot 1 + e^s \right) ds + 1 = \frac{t^4}{6} + te^t + 1. \end{aligned}$$

Other Problem C:

Part 1. Note that $y_2(t) - y_1(t)$ is a continuous function, so if $y_2 - y_1$ is positive at t_2, then $y_2 - y_1$ is positive for t slightly larger than t_2. In particular, there exist values of t between t_2 and t_1 for which $y_2 - y_1$ is positive, and therefore t_2 is not the least upper bound for the set of such values.

If, on the other hand, $y_2 - y_1$ is negative at t_2, then $y_2 - y_1$ is negative for all values of t slightly less than t_2. In particular, t_2 is not the *least* upper bound for the set of $t < t_1$ for which $y_2(t) - y_1(t) \ge 0$. The only remaining possibility is that $y_2(t) - y_1(t) = 0$. □

Part 2. Since $y_1(t_2) = y_2(t_2)$, we have

$$y_1'(t_2) - y_2'(t_2) = f_1(t_2, y_1(t_2)) - f_2(t_2, y_2(t_2)) = f_1(t_2, y_1(t_2)) - f_2(t_2, y_1(t_2)) < 0$$

by hypothesis on f_1 and f_2.

Part 3. Let $h(t) = y_1(t) - y_2(t)$. Since h is C^1 and $h'(t_2) < 0$, we know that $h'(t) < 0$ for t near t_2. Therefore, h is decreasing for t near t_2. In particular, for t slightly greater than t_2, we have $0 < h(t) = y_1(t) - y_2(t) < y_1(t_2) - y_2(t_2)$. That is, $y_1(t) < y_2(t)$. □

Part 4. The fact that $y_1(t) < y_2(t)$ for t slightly larger than t_2 contradicts the assumption that t_2 is an upper bound for the set of such t. This means that t_2 does not exist. This in turn means that there is no t_1 for which $y_1(t_1) > y_2(t_1)$. We conclude that $y_1(t) \le y_2(t)$ for all $t > t_0$. □

Part 5. A similar argument shows that $y_1(t) \ge y_2(t)$ for all $t < t_0$.

Part 6. Suppose that there is actually a point $t_1 > t_0$ for which $y_1(t_1) = y_2(t_1)$. Then, as in part 2, we conclude that $y_1'(t_1) - y_2'(t_1) < 0$. And as in part 3, we conclude that $h(t) = y_1(t) - y_2(t)$ is decreasing for t near t_1. But this means that $y_1(t) > y_2(t)$ for t slightly *less* than t_1. Such t would be greater than t_0, and our earlier arguments have already shown that $y_1(t) \leq y_2(t)$ for all $t \geq t_0$. $\square$