Page 227/ #2:

Part (a). $\limsup a_n = 1$ and $\liminf a_n = 0$.

Part (b). Let $b_n = a_{n+1}/a_n$. Then $b_n = n!$ if $n = 2^k - 1$, $b_n = 1/n!$ if $n = 2^k$, and $b_n = 1/(n+1)$ otherwise. Therefore, $\limsup b_n = \infty$.

Part (c). Let $b_n = a_n^{1/n}$. Then $b_n = 1$ if $n = 2^k$ and $b_n = \sqrt[n]{1/n!}$ otherwise. Since $b_n \leq 1$ for all n and there are arbitrarily large n (i.e. those n equal to 2^k) for which $b_n = 1$, we conclude that $\limsup b_n = 1$.

Page 227/ #6:

$$\begin{aligned}\limsup(a_n + b_n) &= \lim_{n\to\infty}(\sup\{a_j + b_j\}_{j=n}^{\infty}) \leq \lim_{n\to\infty}(\sup\{a_j\}_{j=n}^{\infty} + \sup\{b_j\}_{j=n}^{\infty}) \\ &= \lim_{n\to\infty}(\sup\{a_j\}_{j=n}^{\infty}) + \lim_{n\to\infty}(\sup\{b_j\}_{j=n}^{\infty}) = \limsup a_n + \limsup b_n.\end{aligned}$$

□

To see that the reverse inequality need not hold, try $a_n = 1 - (-1)^n$ and $b_n = (-1)^n$. The sum of the lim sup's is 3, but the lim sup of the sum is 1.

Page 227/ #8:

Let $a = \limsup a_n$. Let $j > 0$ be an integer $\epsilon = 1/j$. Then by Theorem 6.1, there exists $N \in \mathbf{N}$ such that $n \geq N$ implies that

$$a_n - a < \epsilon.$$

Also by Theorem 6.1, there exists *some* $n \geq N$, which we designate n_j such that

$$a_{n_j} - a > -\epsilon.$$

We find such an n_j for each j and have that

$$-1/j = -\epsilon < a_{n_j} - a < \epsilon = 1/j$$

for every j. The squeeze theorem therefore guarantees that $\lim_{j\to\infty} a_{n_j} = a$. Now technically, the n_j's should be increasing for $\{a_{n_j}\}$ to be a subsequence, but this can be arranged by replacing N with $N' = \max\{N, n_1, \ldots, n_{j-1}\}$ when we go to choose n_j. The result is then a subsequence a_{n_j} converging to a as j goes to infinity. □

Page 228/ #9

Again, let $a = \limsup a_n$. Since each limit point of $\{a_n\}$ is the limit of a subsequence, the previous problem shows that $a \in P$. Therefore, $\sup P \geq a$. On the other hand, if $c \in P$, then $c = \lim_{j\to\infty} a_{n_j}$ for some subsequence. And if $\epsilon > 0$, then there exists $N \in \mathbf{N}$ such that $a_n - a < \epsilon$ when $n \geq N$. Since the n_j's are increasing, there exists $J \in \mathbf{N}$ such that $n_j \geq N$ when $j \geq J$. Therefore, $j \geq J$ implies that

$$a_{n_j} - a < \epsilon.$$

Letting j tend to infinity shows that $c - a < \epsilon$. This is true for any $\epsilon > 0$, so in fact, we see that $c - a \leq 0$, i.e. $c \leq a$. Hence a is an upper bound for P and must therefore be at least as large as $\sup P$. We conclude that $a = \sup P$. □