Solutions for Homework 6

Page 236/ #1:

(a) Converges—use the ratio test.

(c) Diverges—terms go to infinity.

(e) Diverges by the comparison test—note that

$$\sqrt{j+1}-\sqrt{j}=\frac{1}{\sqrt{j+1}+\sqrt{j}}\geq\frac{1}{2\sqrt{j+1}}\geq\frac{1}{3\sqrt{j}}\geq\frac{1}{3j}$$

for all $j\geq 1$. We know from the textbook that $\sum\frac{1}{j}$ diverges.

(g) Converges by the root test. Notice

$$\lim_{n\to\infty}|e^{-j+\sin j}|^{1/j}=\lim_{j\to\infty}e^{-1+\frac{\sin j}{j}}=e^{-1}<1.$$

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If we try to apply the ratio test, then we find that $|a_{n+1}/a_n|=2$ if n is odd and $1/2$ if n is even. Thus the limsup of this ratio is 2 (i.e. the limit point obtained by taking the limit of the ratio for odd n), and the ratio test tells us nothing. On the other hand, if we apply the root test, we find

$$\limsup_{j\to\infty}|a_j|^{1/j}=\limsup\frac{1}{2}|2^{(-1)^j}|^{1/j}.$$

But $2^{(-1)^j}$ is always between $1/2$ and 2. Hence

$$\sqrt[j]{\frac{1}{2}}\leq(2^{(-1)^j})^{1/j}\leq\sqrt[j]{2}$$

for all j, and the Sqeeze Theorem allows us to conclude that

$$1=\lim_{j\to\infty}(2^{(-1)^j})^{1/j}=\limsup_{j\to\infty}(2^{(-1)^j})^{1/j}.$$

Plugging this back into the first equation gives

$$\limsup_{j\to\infty}|a_j|^{1/j}=\frac{1}{2},$$

which is less than one. Therefore the series converges.

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By hypothesis, there exists a number M such that $|b_j|\leq M$ for all j. Therefore $|a_jb_j|\leq M|a_j|$ for all j. Also, since $\sum_{j=1}^{\infty}|a_j|$ converges, so does $\sum_{j=1}^{\infty}M|a_j|$ (Theorem 6.2.1, part (d)). The comparison test therefore implies that $\sum_{j=1}^{\infty}|b_ja_j|$ converges. Theorem 6.2.1 part (c) implies in turn that $\sum_{j=1}^{\infty}b_ja_j$ converges. □

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For part (a), consider $a_j=1/n^2$. For part (b), let S_n be the nth partial sum of $\sum_{j=1}^{\infty}\frac{\sqrt{a_j}}{j}$. Since the terms of this series are positive, we see that $\{S_n\}$ is an increasing sequence. Therefore in order to show that the series converges, we need only show that S_n is bounded above.

The Cauchy-Schwartz inequality (page 218, #10) gives us that

$$S_n = \sum_{j=1}^{n} \frac{\sqrt{a_j}}{j} \leq \left(\sum_{j=1}^{n} \frac{1}{j^2} \right)^{1/2} \left(\sum_{j=1}^{n} a_j \right)^{1/2} = (R_n \cdot T_n)^{1/2},$$

where R_n is the nth partial sum for the series $\sum_{j=1}^{\infty} \frac{1}{j^2}$ and T_n is the nth partial sum for $\sum_{j=1}^{\infty} a_j$. These last two series converge, which means by definition that their sequences of partial sums converge. Since convergent sequences are bounded, there exist upper bounds R and T for $\{R_n\}$ and $\{T_n\}$, respectively. We conclude that

$$S_n \leq (RT)^{1/2}$$

for all n and that $\{S_n\}$ is therefore bounded above. This completes the proof. □

Other Problem A.

Part 1. If $k \leq 0$, then $\frac{1}{n(\log n)^k} \geq \frac{1}{n}$ for all n. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the comparison test implies that $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^k}$ diverges, too.

Part 2. Note that

$$S_n = \int_1^{n+1} f(x)\, dx$$

where $f : [3, \infty) \to \mathbf{R}$ is the piecewise constant function equal to $\frac{1}{n(\log n)^k}$ for all $x \in [n, n+1)$. Note further that

$$\frac{1}{x(\log x)^k} \leq f(x) \leq \frac{1}{(x-1)(\log(x-1))^k}$$

for all $x \in [3, \infty)$. If $k = 1$, then integrating this inequality from three to $n+1$ gives us the upper and lower bounds.

$$\log(\log(n+1)) - \log(\log 3) \leq S_n \leq \log(\log n) - \log(\log 2)$$

for all n. If $k > 1$, the bounds become

$$\frac{1}{(k-1)(\log 3)^{k-1}} - \frac{1}{(k-1)(\log(n+1))^{k-1}} \leq S_n \leq \frac{1}{(k-1)(\log 2)^{k-1}} - \frac{1}{(k-1)(\log n)^{k-1}}.$$

Part 3. In all cases, the terms of the series are increasing, so the sequence of partial sums is increasing. Therefore, the series converges if and only if the sequence of partial sums is bounded. When $k = 1$ our lower bound for S_n goes to infinity as n increases, so S_n must diverge to infinity as well—i.e. the series diverges when $k = 1$. When $k > 1$, we apply our upper bound to see that

$$S_n \leq \frac{1}{(k-1)(\log 2)^{k-1}}$$

for all $n \geq 3$. Therefore, the series converges.

Part 4. We again use the fact that $f(x) \leq \frac{1}{(n-1)(\log(n-1))^2}$ for all $x \in [3, \infty)$. Choosing integers $m \geq n \geq 3$ and integrating from n to $m+1$ gives us the bound

$$|S_m - S_n| = S_m - S_n \leq \frac{1}{\log n} - \frac{1}{\log m}$$

Letting m go to infinity then gives

$$|S - S_n| = S - S_n \leq \frac{1}{\log n}$$

for all $n \geq 3$.

Part 5. Using the upper bound from part 4, we see that we would need to choose n large enough that

$$\frac{1}{\log n} \leq .01.$$

That is, we would have to add up around $n \geq e^{100} \approx 2.7 \times 10^{43}$ terms of the sequence.

Part 6. Note that we can repeat Part 4 using the estimate $f(x) \geq \frac{1}{x(\log x)^2}$ for all $x \in [3, \infty)$ and obtain (after some computation) that

$$S - S_n \geq \frac{1}{(\log(n+1))^2}$$

for all $n \geq 3$. Combining this with the bound in Part 4 and rearranging a bit gives

$$S_n + \frac{1}{(\log(n+1))^2} \leq S \leq S_n + \frac{1}{\log n}.$$

Call the upper bound U_n and the lower bound L_n. By the Mean Value Theorem, there exists a number $c \in (n, n+1)$ such that

$$U_n - L_n = \frac{1}{\log n} - \frac{1}{(\log(n+1))^2} = \frac{1}{c(\log c)^2} \leq \frac{1}{n(\log n)^2}.$$

In particular, we find by plugging in different n on the right side that $U_n - L_n < .01$ when $n \geq 15$. Since S is between U_{15} and L_{15}, we can use either quantity to approximate S to within .01. We compute that $U_{15} = 1.07352$. (Incidentally, Mathematica gives a value of 1.06906 for S, so if Mathematica can be trusted, our approximation really is within .01 of being correct.)