Solutions for Homework 7

Page 243/ #8:

Given $n \in \mathbf{N}$, set

$$g_n(x) = \begin{cases} \sin x & \text{if} \quad j = 0 \mod 4 \\ \cos x & \text{if} \quad j = 1 \mod 4 \\ -\sin x & \text{if} \quad j = 2 \mod 4 \\ -\cos x & \text{if} \quad j = 3 \mod 4 \end{cases}$$

Note that $g_n'(x) = g_{n+1}(x)$ for every n and that $|g_n(x)| \leq 1$ for every n and x.

If $1 \leq N < p$, then I claim that for every $n \leq N-1$, the series

$$\sum_{j=1}^{\infty} (2\pi j)^n a_j g_n(2\pi j x).$$

converges uniformly on $\mathbf{R}$ and is equal to $f^{(n)}(x)$. In particular, f is $N-1$ times differentiable. I will prove this claim by induction on n:

When $n = 0$ this amounts merely to showing that

$$\sum_{j=1}^{\infty} a_j \sin(2\pi j x) \tag{1}$$

converges uniformly. The sum is equal to $f(x) = f^{(0)}(x)$ by definition. By hypothesis, we have

$$|a_j \sin(2\pi j x)| \leq |a_j| \leq C/j^p$$

for all $x \in \mathbf{R}$. Furthermore, p is at least two, so $\sum_{j=1}^{\infty} \frac{C}{j^p}$ converges. Hence the Weierstrass M-test implies that the series in equation **??** converges uniformly for all $x \in \mathbf{R}$. This concludes the case $n = 0$.

Now suppose that the claim is true for $n = k$ and consider the case $n = k+1$. Then the series

$$\sum_{j=1}^{\infty} (2\pi j)^k a_j g_k(2\pi j x)$$

converges uniformly on $\mathbf{R}$ and is equal to $f^{(k)}(x)$. Differentiating this series term by term gives us the new series

$$\sum_{j=1}^{\infty} (2\pi j)^{k+1} a_j g_{k+1}(2\pi j x) = \sum_{j=1}^{\infty} (2\pi j)^n a_j g_n(2\pi j x). \tag{2}$$

We have the following upper bounds for the terms of this new series:

$$|(2\pi j)^n a_j g_n(2\pi j x)| \leq (2\pi j)^n |a_j| \leq \frac{(2\pi)^n C}{j^{p-n}}.$$

Now since $n \leq N-1 < p-1$, we have that $p-n > 1$ and that $\sum_{j=1}^{\infty} 1/j^{p-n}$ converges. Hence the Weierstrass M-test tells us that the series (**??**) converges uniformly for all $x \in \mathbf{R}$. By Theorem 6.3.3 then, we conclude that this series is equal to $\frac{d}{dx}\frac{d^{k-1}f}{dx^{k-1}} = \frac{d^k f}{dx^k}$. That is, the claim is true in the case $n = k+1$ and our proof is complete. $\square$

Page 244/ #15:

Note that $e^{-(x-j)^2}/j^2 \leq e^0/j^2 = 1/j^2$ for all $x \in \mathbf{R}$. Since $\sum_{j=1}^{\infty} 1/j^2$ converges, the Weierstrass M-test tells us that

$$f(x) = \sum_{j=1}^{\infty} \frac{e^{-(x-j)^2}}{j^2} \tag{3}$$

converges uniformly on $\mathbf{R}$ to a continuous function $f(x)$.

It remains to show that $\lim_{x\to\pm\infty} f(x) = 0$. We will do this for $x \to \infty$ only, since the case $x \to -\infty$ is similar. Let $S_n(x)$ denote the nth partial sum for the series in (**??**). Let $\epsilon > 0$ be any fixed number. Since $S_n(x)$ converges uniformly to $f(x)$ (by definition of convergence for a series), there exists an $N \in \mathbf{N}$ such that

$$\sum_{j=n+1}^{\infty} \frac{e^{-(x-j)^2}}{j^2} = |f(x) - S_n(x)| < \epsilon$$

whenever $n \geq N$. In particular, this inequality holds if we choose $n = N$.

Note also that, since $S_N(x)$ is a *finite* sum, we have

$$\begin{aligned} 0 \quad &\leq \quad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sum_{j=1}^{N} \frac{e^{-(x-j)^2}}{j^2}(x) + \lim_{x\to\infty} \sum_{j=n+1}^{\infty} \frac{e^{-(x-j)^2}}{j^2} \\ &= \quad \sum_{j=1}^{N} \lim_{x\to\infty} \frac{e^{-(x-j)^2}}{j^2}(x) + \lim_{x\to\infty} \sum_{j=n+1}^{\infty} \frac{e^{-(x-j)^2}}{j^2} \\ &\leq \quad 0 + \epsilon \end{aligned}$$

The only way that this inequality holds for every $\epsilon > 0$ is if we have $\lim_{x\to\infty} f(x) = 0$. □

Page 251/ #3:

If $f(x) = \ln x$, then for $n \geq 1$, $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}$. Hence the power series for $\ln(x)$ centered at $x_0 = 1$ is

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(1)}{j!}(x-1)^j = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j}(x-1)^j.$$

It's not hard to show that $\lim_{n\to\infty}(1/j)^{1/j} = 1$, so the radius of convergence for this series is 1. Now if $S_n(x)$ is the nth partial sum for the series, then Taylor's theorem tells us that for some c between x and 1, we have

$$f(x) - S_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-1)^{n+1} = \frac{(-1)^n(x-1)^{n+1}}{(n+1)c^{n+1}}.$$

Now if $|x - 1| < 1/2$, we have that $-1/2 < c < 3/2$, so

$$|f(x) - S_n(x)| \leq \frac{(1/2)^{n+1}}{(n+1)(1/2)^{n+1}} = \frac{1}{n+1},$$

In other words, for every $n \in \mathbf{N}$,

$$-\frac{1}{n+1} \leq f(x) - S_n(x) \leq \frac{1}{n+1}$$

By the Squeeze Theorem, we conclude that $\lim_{n\to\infty} |f(x) - S_n(x)| = 0$. □

Page 251/ #8: Since $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$, and the series on the right side converges uniformly on any finite interval, we have $e^{-x^2} = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{j!}$ where the series (again) converges uniformly on any finite interval. Hence we can integrate the latter series term by term (see Theorem 6.3.2) to obtain

$$F(x) = \int e^{-x^2}\, dx = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{j!(2j+1)}.$$

Page 251/ #11:

The radius of convergence is

$$R = (\limsup (j+1)^{1/j}(j+2)^{1/j})^{-1} = 1.$$

If we call the series $f(x)$, then the antiderivative of the antiderivative of f can be obtained by integrating the series twice term by term (Theorem 6.3.2 again, along with the fact that the series converges uniformly on $(-r, r)$ for any $r < R = 1$.) Hence the antiderivative of the antiderivative of f is given by the geometric series

$$F(x) = \sum_{j=0}^{\infty} x^{j+2} = x^2 \sum_{j=0}^{\infty} x^j = \frac{x^2}{1-x}.$$

Now we differentiate twice to get back to f, arriving at the formula

$$f(x) = F''(x) = \frac{2}{1-x}^{3}.$$

$\square$

Page 251/ #12:

Consider the series $\sum_{j=1}^{\infty} \frac{x^j}{j^p}$. This series has radius of convergence equal to one for every p and therefore converges when $|x| < 1$ and diverges when $|x| > 1$. However, whether the series converges or diverges at the points $x = 1$ and $x = -1$ depends on p. At $x = 1$, we have

$$\sum_{j=1}^{\infty} \frac{x^j}{j^p} = \sum_{j=1}^{\infty} \frac{1}{j^p}$$

and at $x = -1$, we have

$$\sum_{j=1}^{\infty} \frac{x^j}{j^p} = \sum_{j=1}^{\infty} \frac{(-1)^j}{j^p}$$

If $p = 0$, then both of these series diverge, because the terms don't go to zero in either of them. If $p = 1$, then the first series is the harmonic series and therefore diverges, whereas the second series is the alternating harmonic series (example from class) and therefore converges. If $p = 2$, then the first series converges, and this means that the second series converges absolutely. Taken together, these examples show that a power series can converge at both endpoints of its interval of convergence, diverge at both endpoints, or converge at one endpoint, but diverge at the other.