Solutions for Homework 8

Problem 1: Compute the operator norm of the matrix

$$\begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix}.$$

(Hint: a vector in $\mathbf{R}^2$ can be written in polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$).

Solution: Consider a vector $(x, y) = (r\cos\theta, r\sin\theta)$ in ^{2}. Note that $||(x, y)|| = r$. Define a new vector (x', y') by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3r\cos\theta + 5r\sin\theta \\ 5r\cos\theta + 3r\sin\theta \end{pmatrix}$$

The operator norm of the matrix is the maximum value of the quantity

$$\frac{||(x', y')||}{||(x, y)||} = \sqrt{34 + 60\cos\theta\sin\theta} = \sqrt{34 + 30\sin 2\theta}$$

The expression under the square root is largest when $\sin 2\theta = 1$ so the operator norm of the matrix is $\sqrt{64} = 8$.

Problem 2: Show *using the definition of limit* that the function $L : \mathbf{R^2} \to \mathbf{R^3}$ given by

$$L(x, y) = \begin{pmatrix} 1 & -2 \\ e & 1 \\ 1 & \pi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

is continuous.

Solution: Let A denote the 3×2 matrix in the definition of L. From class we know that the operator norm of A is no greater than $Cmn = 6\pi$ where C is the maximum of the absolute values of the entries of A and m and n are the number of rows and columns, respectively, of A.

Let $\mathbf{a} \in \mathbf{R^n}$ and $\epsilon > 0$ be given and choose $\delta = \epsilon/6\pi$. Then if $||\mathbf{x} - \mathbf{a}|| < \delta$, we have

$$||L(\mathbf{x}) - L(\mathbf{a})|| = ||A \cdot (\mathbf{x} - \mathbf{a})|| \leq 6\pi||\mathbf{x} - \mathbf{a}|| < 6\pi\delta = \epsilon.$$

Therefore L is continuous at $\mathbf{a}$. Since $\mathbf{a}$ was arbitrary, L is continuous on all of $\mathbf{R}^n$. $\square$

Problem 3: Show that the intersection of finitely many open sets is open. Give an example of a countable collection of open sets $U_j \subset \mathbf{R}^2$, $j \in \mathbf{N}$ whose intersection $\bigcap_{j=1}^{\infty} U_j$ is not open.

Solution: Suppose that $U_1, \ldots, U_k \subset \mathbf{R}^n$ are open sets and let $U = U_1 \cap \cdots \cap U_k$ be their intersection. Then if $\mathbf{a} \in U$, we must show that $\mathbf{a}$ is an interior point of U. To do this, we point out that $\mathbf{a} \in U$ implies that $\mathbf{a} \in U_j$ for $j = 1, \ldots, k$. But each U_j is open, so there exists a number $r_j > 0$ such that $B_{r_j}(\mathbf{a}) \subset U_j$. It follows that the number $r = \min\{r_1, \ldots, r_k\}$ is positive and that $B_r(\mathbf{a}) \subset U_j$ for every $j = 1, \ldots, k$. Therefore $B_r(\mathbf{a}) \subset U$, too. We conclude that $\mathbf{a}$ *is* an interior point of U and that U is open. $\square$

Problem 4: Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a continuous function and let $U \subset \mathbf{R^m}$ be an open set. Show that the set

$$f^{-1}(U)\{\mathbf{x} \in \mathbf{R}^n : f(\mathbf{x}) \in U\}$$

is open.

Solution: Let $U \subset^n$ be open and $W = f^{-1}(U)$. We must show that any point $\mathbf{a} \in W$ is an interior point. To do this, we note that by definition of W, $f(\mathbf{a}) \in U$. Since U is open, we know that there exists an $\epsilon > 0$ such that $B_\epsilon(f(\mathbf{a})) \subset U$. Since f is continuous, we know that there exists $\delta > 0$ such that $||\mathbf{x} - \mathbf{a}|| < \delta$ implies that $||f(\mathbf{x}) - f(\mathbf{a})|| < \epsilon$. In other words $x \in B_\delta(\mathbf{a})$ implies that $f(\mathbf{x}) \in B_\epsilon(f(\mathbf{a}))$–in particular, that $f(\mathbf{x}) \in U$. Therefore $B_\delta(\mathbf{a}) \subset f^{-1}(U) = W$. This shows that $\mathbf{a}$ is an interior point of W and, since $\mathbf{a}$ was chosen arbitrarily, that W is open. □

Problem 5: Let $f, g : \mathbf{R}^n \to \mathbf{R}^m$ be functions and $\mathbf{a} \in^n$, be a point at which both $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})$ exist. Show that $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) + g(\mathbf{x})$ exists and is equal to $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})$.

Solution: Let $\mathbf{a} \in^n$ and $\epsilon > 0$ be given. If we set $\epsilon' = \epsilon/2 > 0$, then (since f and g are continuous) there exist numbers $\delta_1, \delta_2 > 0$ such that $||\mathbf{x} - \mathbf{a}|| < \delta_1$ implies $||f(\mathbf{x}) - f(\mathbf{a})|| < \epsilon'$ and $||\mathbf{x} - \mathbf{a}|| < \delta_2$ implies $||g(\mathbf{x}) - g(\mathbf{a})|| < \epsilon'$. So if we set $\delta = \min\{\delta_1, \delta_2\} > 0$, we see that $||\mathbf{x} - \mathbf{a}|| < \delta$ implies that

$$||(f+g)(\mathbf{x}) + (f+g)(\mathbf{a})|| \leq ||f(\mathbf{x}) - f(\mathbf{a})|| + ||g(\mathbf{x}) - g(\mathbf{a})|| \leq \epsilon' + \epsilon' = \epsilon.$$

Therefore $f + g$ is continuous at $\mathbf{a}$. □

Problem 6: # 2 on page 244 of Wade's book—no proofs necessary.

a: $\lim_{x\to 0} \lim_{y\to 0} f(x,y) = \lim_{y\to 0} \lim_{x\to 0} f(x,y) = 0$. However, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, because if $(x_n, y_n) = (1/n, 1/n)$ then

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} \frac{1}{2} n^2 (\sin(1/n))^2 = \frac{1}{2},$$

which is different than either of the iterated limits.

b: $\lim_{x\to 0} \lim_{y\to 0} f(x,y) = 1$ and $\lim_{y\to 0} \lim_{x\to 0} f(x,y) = 0 = 1/2$. Again, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

c: $\lim_{x\to 0} \lim_{y\to 0} f(x,y) = \lim_{y\to 0} \lim_{x\to 0} f(x,y) = 0$. Moreover, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ because

$$\begin{aligned} |f(x,y)| &= \left| \frac{x-y}{(x^2+y^2)^\alpha} \right| \\ &\leq \frac{|x|}{(x^2+y^2)^\alpha} + \frac{|y|}{(x^2+y^2)^\alpha} \\ &\leq \frac{|x|}{|x|^{2\alpha}} + \frac{|y|}{|y|^{2\alpha}} = |x|^{1-2\alpha} + |y|^{1-2\alpha}, \end{aligned}$$

and since $\alpha < 1/2$, the last expression tends to zero as $(x,y) \to (0,0)$.