Solutions for Homework 9

From Wade's book:

- Pages 250-252: 2, 3, 6, 8
- Pages 259-260: 2 (∂E—the 'boundary of E' is $\overline{E} - \overset{\circ}{E}$), 5, 7

Page 251/ #2: Solution

Let $\mathbf{a} = (x, y)$ and $\mathbf{h} = (h_1, h_2)$ be points in $\mathbf{R^2}$. Let

$$T(\mathbf{h}) = \begin{pmatrix} y & x \\ 1 & 1 \\ 2x & -2y \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

Then

$$f(\mathbf{a} + h) - f(\mathbf{a}) - T(\mathbf{h}) = (h_1 h_2, 0, h_1^2 - h_2^2).$$

Therefore,

$$\begin{aligned} \lim_{\mathbf{h}\to\mathbf{0}} \frac{||f(\mathbf{a}+h) - f(\mathbf{a}) - T(\mathbf{h})||}{||\mathbf{h}||} &\leq \lim_{\mathbf{h}\to\mathbf{0}} \frac{|h_1 h_2|}{||\mathbf{h}||} + \frac{|h_1^2|}{||\mathbf{h}||} + \frac{|h_2^2|}{||\mathbf{h}||} \\ &\leq \lim_{\mathbf{h}\to\mathbf{0}} \frac{|h_1 h_2|}{|h_1|} + \frac{|h_1^2|}{|h_1|} + \frac{|h_2^2|}{|h_2|} = 0. \end{aligned}$$

Therefore f is differentiable at $\mathbf{a}$, and $Df_{\mathbf{a}}(\mathbf{h}) = T(\mathbf{h})$.

Page 251/ #6a: Solution

Suppose that f and g are differentiable at $\mathbf{a}$. Then

$$\begin{aligned} 0 &\leq \lim_{\mathbf{h}\to 0} \frac{||(f+g)(\mathbf{a}+h) - (f+g)(\mathbf{a}) - Df_{\mathbf{a}}(\mathbf{h}) - Dg_{\mathbf{a}}(\mathbf{h})||}{||\mathbf{h}||} \\ &\leq \lim_{\mathbf{h}\to 0} \frac{||f(\mathbf{a}+h) - f(\mathbf{a}) - Df_{\mathbf{a}}(\mathbf{h})||}{||\mathbf{h}||} + \frac{||g(\mathbf{a}+h) - g(\mathbf{a}) - Dg_{\mathbf{a}}(\mathbf{h})||}{||\mathbf{h}||} = 0. \end{aligned}$$

That is, $f + g$ is differentiable at $\mathbf{a}$, and $D(f+g)_{\mathbf{a}} = Df_{\mathbf{a}} + Dg_{\mathbf{a}}$. □

Page 251/ #8: Solution

Since T is linear,

$$T(\mathbf{a} + \mathbf{h}) - T(\mathbf{a}) - T(\mathbf{h}) = T(\mathbf{a} + \mathbf{h} - \mathbf{a} - \mathbf{h}) = T(\mathbf{0}) = \mathbf{0}$$

for every $\mathbf{a}, \mathbf{h} \in^n$. Hence,

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{||T(\mathbf{a}+\mathbf{h}) - T(\mathbf{a}) - T(\mathbf{h})||}{||\mathbf{h}||} = 0.$$

Therefore, T is differentiable at $\mathbf{a}$, and $DT_{\mathbf{a}}(\mathbf{h}) = T(\mathbf{h})$ for every $\mathbf{a} \in^n$. □

Page 260/ #2: Solution

a: $\overline{E} = E$, $\overset{\circ}{E} = \{(x,y) \in^2 : x^2 + 4y^2 < 1\}$, and $\partial E = \{(x,y) \in^2 : x^2 + 4y^2 = 1\}$

b: $\overline{E} = E = \partial E$, $\overset{\circ}{E} = \emptyset$.

c: $\overline{E} = \{(x,y) \in^2 : y \geq x^2, 0 \leq y \leq 1\}$, $\overset{\circ}{E} = \{(x,y) \in^2 : y > x^2, 0 < y < 1\}$, $\partial E = \{(x,y) \in^2 : y = x^2, 0 < y < 1\} \bigcup \{(x,y) \in^2 : y = 1, -1 \leq x \leq 1\}$.

Page 260/ #5: Solution

Suppose that $\inf_{\mathbf{x}\in E} ||\mathbf{x}-\mathbf{a}|| = 0$. Then for every $j \in$, there exists $\mathbf{x}^j$ such that $||\mathbf{x}^j - \mathbf{a}|| < 1/j$. But if this is true, then

$$\lim_{j\to\infty} ||\mathbf{x}^j - \mathbf{a}|| \leq \lim_{j\to\infty} 1/j = 0.$$

In other words $\mathbf{x}^j \to \mathbf{a}$, so $\mathbf{a}$ is a limit point of E. Since E is closed, we must have that $\mathbf{a} \in E$. This contradicts the hypothesis that $\mathbf{a} \notin E$. Hence our initial supposition was wrong, and we conclude that $\inf_{\mathbf{x}\in E} ||\mathbf{x} - \mathbf{a}|| > 0$. $\square$

Page 260/ #7: Solution

a: $A = [0, 1]$, $B = [1, 2]$.
b: $A = (0, 1)$, $B = (1, 2)$.
c: $A = [0, 1]$, $B = [1, 2]$ for the first part and $A = (0, 1)$, $B = (1, 2)$ for the second part. (Note that there's a typo in the statement of the problem. The last "$\partial A \cup \partial B$" should be "$\partial A \cap \partial B$" instead.)