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Math 361 Exam 1; Mon Oct 13, 10:40–11:30am

Instructions. Answer questions 1–6. You must show all necessary working to receive full points, unless otherwise indicated.

1. *(10 points)* Let $G = S_4$ denote the symmetric group $A(\{1,2,3,4\})$ on 4 letters. Define the elements $g = \binom{1\,2\,3\,4}{2\,3\,4\,1}$ and $h = \binom{1\,2\,3\,4}{3\,2\,1\,4}$ in G.

(a) Calculate $g^{-1}h$.

(b) List the elements of the cyclic group generated by g. What is the order of g?

2. *(20 points)*

(a) Give a careful definition of the term "group."

(b) State whether the following are groups or not (no working is required): (1) the set of real numbers with binary operation $*$ given by $a * b = 2(a+b)$ (2) the set of all positive real numbers under multiplication (3) the set of all non-negative real numbers under addition.

3. *(10 points)* Let G be a group.

(a) State formulae for $(ab)^{-1}$ and $(a^{-1})^{-1}$, for $a, b \in G$.

(b) For $x, y \in G$, define the "commutator" $[x, y]$ of x and y by $[x, y] := xyx^{-1}y^{-1}$. Show by direct calculation, using the formulae from (a), that if $x, y, z \in G$, then

$$[zxz^{-1}, zyz^{-1}] = z[x, y]z^{-1}.$$

4. *(20 points)* Let G be the group of non-zero real numbers under multiplication. (You do NOT have to prove that G is a group).

(a) Prove that $H = \{\, 2^n \mid n \in Z \,\}$ is a subgroup of G.

(b) Let n be a positive integer. Prove that the function $f\colon G \to G$ given by $f(x) = x^n$ is a group homomorphism.

(c) Find the kernel and image of f (your answer to this part will depend on whether n is even or odd). For which values of n is f an isomorphism?

5. *(20 points)* (a) Explain what is meant by a right coset Hg of a subgroup H of a group G. Define the index $[G\colon H]$ of H in G.

(b) State Lagrange's theorem, and explain briefly how cosets are involved in its proof.

(c) Is it possible for a group of order 75 to have (1) a subgroup of order 50? (2) an element of order 15? (3) a subgroup of index 10? (Simply state yes or no for each part; no working is required).

6. *(20 points)* Let $G = Z_{19}^*$ be the group of units of the integers mod 19, consisting of congruence classes $[n]$ modulo 19 with gcd$(n, 19)$=1, under the multiplication $[n][m] = [nm]$. (You do NOT have to prove G is a group).

(a) Determine whether G is a cyclic group.

(b) List the 6 elements of the cyclic subgroup H of G generated by [8].

(c) Find all the distinct right cosets of H in G. What is the index $[G : H]$?

(d) Explain briefly why H is a normal subgroup of G. Write down the multiplication table for the quotient group G/H.