Math 361 Exam 2; Mon Nov 24, 1997; 10:40–11:30am

Instructions. Answer questions 1–4. You must show all necessary working to receive full points for a problem.

1. *(25 points)* Let S_6 denote the symmetric group on six letters. Let $g = (4,3,5)(4,3,2)(6,5) \in S_6$.

(a) Express g as a product of disjoint cycles.

(b) Express g as a product of transpositions (two-cycles).

(c) Is g even or odd? For which integers n is g^n an element of the alternating group A_6?

(d) Compute the order of g.

(e) Find an element h of S_6 such that $hgh^{-1} = (1,6)(2,3,4)$.

2. *(25 points)*

(a) Calculate $(2+i-3j)(1-i+j)^{-1}$ in the ring of quaternions.

(b) In each part (i)–(iv), either give an example of a ring R satisfying the indicated condition or prove there is no such ring: (i) R is commutative but not an integral domain (ii) R is an integral domain but not a field (iii) R is a field but not an integral domain (iv) R is a division ring but not a field.

3. *(30 points)*

(a) For a ring R, explain carefully what is meant by saying that I is an ideal of R.

(b) Check that $I = \{[0],[3]\}$ is an ideal of the ring $R = Z_6$. List the distinct cosets of I in R and write down the multiplication and addition tables for the quotient ring R/I.

(c) State the isomorphism theorem giving the relationship between S, T and the kernel of a surjective ring homomorphism $\theta\colon S \to T$.

(d) Describe explicitly a surjective ring homomorphism $\theta\colon Z_6 \to Z_3$ with $\ker\theta = I$, and conclude that there is a ring isomomorphism

$$Z_6/I \cong Z_3.$$

(e) List all the ideals J of Z_6 (there are four of them including I) and decide which are maximal ideals.

4. *(20 points)* Suppose that R is a "Boolean ring" i.e. a ring such that $x^2 = x$ for all $x \in R$.

(a) By considering $(a+1)^2$, show that $a = -a$ for every $a \in R$.

(b) Prove that R is commutative by considering $(c+d)^2$ for $c, d \in R$ and using (a).

(c) Explain what is meant by the characteristic of a commutative ring. What is the characteristic of R?

(d) Give an example of a Boolean ring.