Math 361 Final Exam; Tues Dec 16, 1997 8:00–10:00am

Instructions. Answer questions 1–5. You must show all necessary working to receive full points for a problem.

1. *(30 points)*

(a) Find all the rational numbers which are roots of the polynomial $X^4 + 2X^3 + X + 2$, and hence factor this polynomial into a product of monic irreducible polynomials in $Q[X]$.

(b) State Eisenstein's criterion and use it to show that the polynomial

$$f(X) = 20X^5 - 12X^4 + 15X^3 - 63X^2 + 30X - 60$$

is irreducible over the rational numbers.

(c) Let $g = (1, 2)(3, 4, 5, 6)$ and $h = (6, 5)(4, 3, 2, 1)$ in the symmetric group S_6 on six letters. Find an element $x \in S_6$ so $g = xhx^{-1}$.

2. *(30 points)* Let G be a group and H be a subgroup of G.

(a) Carefully explain what is meant by the left coset gH of an element g of G.

(b) Define the index $[G\colon H]$ of H in G.

(c) Suppose that G is finite. Define the order $|G|$ of G. What is the relationship between the orders of G and H, and the index $[G\colon H]$?

(d) Define the order of an element $g \in G$. Use your answer to (c) to show that if G is a finite group, the order of g divides the order of G.

3. *(30 points)* In this question, G denotes the subset $G = \{\pm 1, \pm i, \pm j, \pm k\}$ of 8 elements of the quaternions. You may ASSUME that G is a group under the usual multiplication of quaternions.

(a) Show that $N = \{\pm 1\}$ is a normal subgroup of G.

(b) Explicitly list the elements of the cosets gN of N in G.

(c) Write down the multiplication table for the quotient group G/N.

(d) Show that G/N is isomorphic to $Z_2 \times Z_2$ by explicitly describing an isomorphism between these two groups.

4. *(30 points)* Let R be a ring, and I, J be two ideals of R.

(a) Define the terms ring and ideal.

(b) Show carefully that $I \cap J$ is an ideal of R, and that if $x \in I$ and $y \in J$, then $xy \in I \cap J$.

(c) Explain what is meant by saying that R is an integral domain.

(d) Suppose that R is an integral domain and $I \cap J = \{0\}$. Use your answer to (b) to show that either $I = \{0\}$ or $J = \{0\}$.

5. *(30 points)* Let $R = Z_2[X]$ denote the ring of polynomials in the variable X over the field of 2 elements. Set $a(X) = X^4 + X + 1 \in R$ and $b(X) = X^2 + 1 \in R$.

(a) Prove that $a(X)$ is an irreducible polynomial in R. Is $b(X)$ irreducible?

(b) Find explicitly polynomials $f(X)$ and $g(X)$ so that $f(X)b(X) + g(x)a(X) = 1$.

(c) Explicitly list the distinct elements of the ring R/I where $I = (a(X))$ is the principal ideal generated by $a(X)$. Explain why R/I is a field. How many elements does R/I have?

(d) In R/I, prove that $(b(X) + I)^2 = X + I$, and calculate $(b(X) + I)^{-1}$ explicitly.