Final for Honors Algebra III, Math 361

Name: December 15, 1998

Instructions: The test is 120 minutes in length.

1. *(30 points)* Consider the set $\mathbb{Z}^3$ of three vectors with integer coefficients. Let

$$H := \mathrm{rowsp}_{\mathbb{Z}} \begin{pmatrix} 5 & 4 & 2 \\ -6 & 11 & 6 \\ -13 & 0 & 13 \end{pmatrix}.$$

(a) Show that $\mathbb{Z}^3$ is a group with respect to vector addition and H is a subgroup.

(b) Consider the factor group $\mathbb{Z}^3/H$. Determine this factor group as good as you can. Determine for example: Is the group infinite or finite. If it is finite how many elements does it have? Is the group Abelian or not? Is the group cyclic or not?

2. *(30 points)* Let $\mathbb{Z}_2$ be the field of two elements.

$$G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}_2 \text{ and } ad - bc = 1 \right\}.$$

(a) Show that G forms a group with respect to matrix multiplication.

(b) How many elements does G have?

(c) Show that G is isomorphic to a symmetric group by constructing an explicit isomorphism.

3. *(30 points)*

(a) Define the alternating group A_n.

(b) If $n \geq 3$ show that every element is a product of 3-cycles.

(c) Find a normal subgroup in A_4 of order 4.

4. *(30 points)* Let $R = \mathbb{C}[x]$ be the polynomial ring with coefficients in the complex numbers $\mathbb{C}$.

(a) If $\varphi(x) \in \mathbb{C}[x]$ is a polynomial define what it means: "z is a root of $\varphi(x)$".

(b) If $\varphi(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n show that $\varphi(x)$ can have at most n roots in $\mathbb{C}$.

(c) Show that $\varphi(x) \in \mathbb{C}[x]$ has a double root at z if and only if $\frac{d}{dx}(\varphi(z)) = \varphi'(z) = 0$.

(d) Let

$$M = \{f \in \mathbb{C}[x] \mid f(1) = 0\}.$$

Show that M is a maximal ideal of $\mathbb{C}[x]$. What can you say about $\mathbb{C}[x]/M$?

5. *(30 points)* Let $R = \mathbb{Z}_2[x]$ denote the ring of polynomials in the variable x over the field of 2 elements. Set $f(x) = x^4 + x + 1 \in R$ and $g(x) = x^2 + 1 \in R$.

(a) Prove that $f(x)$ is an irreducible polynomial in R. Is $g(x)$ irreducible?

(b) Find explicitly polynomials $a(x)$ and $b(x)$ so that $a(x)f(x) + b(x)g(x) = 1$.

(c) Explicitly list the distinct elements of the ring R/I where $I = (f(x))$ is the principal ideal generated by $f(x)$. Explain why R/I is a field. How many elements does R/I have?

(d) In R/I, prove that $(g(x) + I)^2 = x + I$, and calculate $(g(x) + I)^{-1}$ explicitly.