MATHEMATICS 361 Name ___________________

Final Examination

December 15, 1999 6 questions

Note: A proof is needed in an answer only when the word "prove" appears in the question.

Z and **Q** denote the ring of integers and the field of rational numbers.

1 *(21 points)* Suppose a , b are integers, not both 0 .

(a) Give a careful definition of the greatest common divisor c of a and b.

(b) Prove that c is equal to the smallest positive integer in the set

$$\{ma + nb \mid m, n \in \mathbf{Z}\} .$$

(c) Prove that, if $d \in \mathbf{Z}$ and a divides bd , then a divides cd .

2 *(30 points)* Let S be the set of all ordered pairs (a,b), where a and b are non-negative integers. Define $(a,b) \sim (c,d)$ to mean that $a + d = c + b$.

(a) Prove that $\sim$ is an equivalence relation on S.

(b) Let $[a,b]$ denote the equivalence class of (a,b). How is this defined?

(c) Let T be the set of all equivalence classes, and let the map $\theta : T \to \mathbf{Z}$ be given by

$$\theta([a,b]) = a - b .$$

Prove that θ is a well-defined map.

(d) Prove that θ is a bijection.

3 *(21 points)* Suppose H is a subgroup of a group G.

(a) Define what it means for H to be normal in G.

(b) If H is normal in G, explain what the elements of the quotient group G/H are and how they are multiplied, and give the definition of the natural homomorphism $\theta : G \rightarrow G/H$.

(c) Give the definition of the kernel of a group homomorphism, and prove that the kernel of the homomorphism θ in (b) is equal to H.

4 *(30 points)* Let R , R' be rings, and $\theta : R \to R'$ a surjective homomorphism.

(a) Define the kernel of θ , and prove that it is an ideal of R .

(b) Explain the connection between the ideals of R' and the ideals of R .

(c) If R' is commutative (and has a unit element), give a condition regarding ideals of R' under which R' will be a field. How does this translate into a condition regarding ideals of R ?

5 *(24 points)* Let S_n and A_n be the symmetric and alternating groups of degree n , $n > 1$.

(a) Define what it means for an element of S_n to be a cycle of length k .

(b) Given an expression of an element σ of S_n as a product of disjoint cycles, explain how to find the order of σ , and how to determine whether or not σ lies in A_n .

(c) Prove that S_5 has an element of order 6 , but A_5 does not.

6 *(24 points)* Suppose $f(x)$, $p(x)$ are non-zero polynomials in $\mathbb{Q}[x]$.

(a) Give a condition involving an ideal or ideals, which is equivalent with $p(x)$ dividing $f(x)$ in $\mathbb{Q}[x]$.

(b) Give a condition involving an ideal, which is equivalent with $p(x)$ being an irreducible polynomial in $\mathbb{Q}[x]$.

For the rest of the question, let $f(x) = 2x^4 - 3x^3 + x^2 + 4x - 2$.

(c) Find all the rational roots of $f(x)$.

(d) Factor $f(x)$ into a product of irreducible polynomials in $\mathbb{Q}[x]$.

(e) Find all the maximal ideals of $\mathbb{Q}[x]$ which contain $f(x)$.