MATHEMATICS 361 **Name** ____________________

Final Examination

December 14, 2000 6 questions

1. (*30 points*) Suppose S, T are sets, $\sim$ is an equivalence relation on S, and $f : T \to S$ is a mapping.

(a) Define what it means to say that $\sim$ is an equivalence relation on S.

(b) Define the equivalence class $[a]$ of an element a of S.

(c) If $x, y \in T$, define $x \approx y$ to mean that $f(x) \sim f(y)$. Show that $\approx$ is an equivalence relation on T.

(d) If C is an equivalence class in T, show that $f(C) = \{f(x) | x \in C\}$ is a subset of a single equivalence class C' in S.

2. (*30 points*) Let m, n be integers (not both 0), and let x be an element of a group G.

(a) Define carefully what it means for m and n to be relatively prime.

(b) It is true that the following statements are equivalent.

(i) m and n are relatively prime.
(ii) There exist integers a, b such that $am + bn = 1$.

Give a proof that (ii) $\Rightarrow$ (i).

(c) Explain what is meant by the cyclic subgroup of G generated by x.

(d) If x has order n, show that x lies in the subgroup of G generated by x^m, if and only if, m and n are relatively prime.

3. (*20 points*) Which of the following is a subgroup of the additive group of the real numbers? [*No justification needed.*]

(a) The set of all positive real numbers.

(b) The set of all rational numbers.

(c) The set of all irrational numbers.

(d) The closed interval [-1,1].

(e) The set of all numbers of the form $a + b\pi + c\pi^2$, where a, b, c are integers.

4. (*30 points*) Let G, G' be groups, $\theta : G \to G'$ a homomorphism, and K the kernel of θ.

(a) Define what it means to say that θ is a homomorphism.

(b) Define what it means to say that K is the kernel of θ.

(c) Show that K is a normal subgroup of G.

(d) If $a, b \in G, ab^{-1} \in K$, and n is an integer, show that $a^n b^{-n} \in K$.

5. (*20 points*) Let R be a ring (with unit).

(a) Define what it means for a subset I of R to be a left ideal in R.

(b) Suppose A is a subset of R with the property that

$$r \in R, a \in A \Rightarrow ra \in A.$$

Let I be the set of all finite sums $a_1 + a_2 + ... + a_m$, where all $a_i \in A$. Prove that I is a left ideal of R.

6. (*20 points*) Let $F[x]$ be the polynomial ring in x over a field F, $p(x) \in F[x]$, $\deg p(x) > 0$. Let $I = (p(x))$ be the principal ideal of $F[x]$ generated by $p(x)$.

(a) Define what it means to say that $p(x)$ is irreducible over F.

(b) Define what it means to say that I is a maximal ideal of $F[x]$.

(c) State two theorems of the course which can be used to show that $p(x)$ is irreducible over F if, and only if, $F[x]/I$ is a field.