MATHEMATICS 361 Name ___________________

Test 1

September 15, 2000

1 (25 points)

Give careful definitions of

(a) the greatest common divisor c of two integers a , b (not both 0)

(b) a group

(c) a subgroup of a group G

(d) the centralizer in a group G of an element a of G

(e) an equivalence relation on a set S

2 (18 points)

Let $f : S \to T$ be a mapping. If X is any subset of S, $f(X)$ denotes the image of X under f. Let A, B be subsets of S.

(a) Show that $f(A \cup B) = f(A) \cup f(B)$.

(b) Show that $f(A \cap B) \subseteq f(A) \cap f(B)$.

(c) Give an example in which equality does not hold in (b).

3 (14 points)

Let S be a set, $A(S)$ the group of all bijective (i.e., one-to-one and onto) maps of S on itself.
Let $f, g \in A(S)$, and suppose that $fg = gf$.
Let $T = \{s \in S \mid g(s) = s\}$.

(a) Show that, if $t \in T$, then $f(t) \in T$ and $f^{-1}(t) \in T$.

(b) Show that f maps T on itself, i.e., $f(T) = T$.

4 (17 points)

Let G be the set of all ordered pairs (a,b), where a and b are integers, provided with a multiplication given by

$$(a,b)(c,d) = (a + c,\ b + (-1)^a d)\ .$$

Then G is a group (you don't have to prove this).

(a) What element of G is the identity element?

(b) For an element (a,b), find the inverse $(a,b)^{-1}$ in G.

(c) Find the centralizer in G of the element $(0,1)$.

5 (26 points)

Let G be a group with identity element e. For a in G, define the maps

$$T_a : G \to G , \quad T_a(x) = ax ,$$

$$T_a' : G \to G , \quad T_a'(x) = xa .$$

Show each of the following statements, with each step of your argument using just one of the axioms for a group (you don't have to state which one).

(a) T_a is a one-to-one map.

(b) T_a is an onto map.

(c) $T_a(e) = a$.

(d) $T_a \circ T_b' = T_b' \circ T_a$, for all a and b in G .

(e) If $f : G \to G$ is any map such that $f \circ T_b' = T_b' \circ f$, for every b in G , then there exists an element a in G such that $f = T_a$. [*Hint*: part (c)]