MATHEMATICS 361 Name ____________________

Test 3

November 15, 2000 4 questions

1 (20 points)

(a) Give a careful definition of the (external) direct product $G_1 \times G_2$ of two groups G_1, G_2 . [*You do not need to verify that it satisfies the group axioms.*]

(b) If H , K are normal subgroups of a group G , and

$$\theta : G \to G/H \times G/K$$

is the mapping given by $\theta(x) = (xH, xK)$,

show that θ is a homomorphism, and prove that its kernel is $H \cap K$.

(c) If H , K are normal subgroups of a group G , show that $H \cap K$ is a normal subgroup of G , and that $G/(H \cap K)$ is isomorphic to a subgroup of $G/H \times G/K$.

2 (25 points)

(a) Define what is meant by the cycle $(a_1 \ \ a_2 \ \ . . . \ a_k)$ in the symmetric group S_n .

(b) Explain what is meant by saying that an element σ of S_n is an even permutation or an odd permutation. [*No proof needed.*]

(c) Express the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 7 & 1 & 2 & 6 & 8 & 3 \end{pmatrix}$ as a product of disjoint cycles, and determine whether it is even or odd.

3 (25 points)

Let H be the division ring of real quaternions, with the usual elements i, j, k, satisfying the relations

$$i^2 = j^2 = k^2 = -1 \text{ , } ij = k = -ji \text{ , etc.}$$

Let α_0, α_1, α_2, α_3 be real numbers.

(a) Compute $(\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k)^2$.

(b) Show that $(\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k)^2 = -1$ if, and only if,

$$\alpha_0 = 0 \text{ , and } \alpha_1{}^2 + \alpha_2{}^2 + \alpha_3{}^2 = 1 \text{ .}$$

(c) Deduce that the equation $x^2 = -1$ has infinitely many solutions in H .

4 (30 points)

Let Let R be the ring $M_2(Z)$ of all 2×2 matrices over Z, with the usual matrix addition and multiplication. Let S, T be the subsets given by

$$S = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a, b, c \in Z \right\}, \quad T = \left\{ \begin{pmatrix} 0 & c \\ 0 & b \end{pmatrix} \mid b, c \in Z \right\},$$

and let $\theta : S \to Z$ be the mapping given by

$$\theta \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} = a .$$

Show that

(a) S is a subring of R (containing the identity element).

(b) θ is a ring homomorphism of S onto Z.

(c) T is a two-sided ideal of S, and S/T is isomorphic to Z.