Math 362 Exam 1; Mon, Feb24, 1997 12:50–1:40pm

Instructions. Answer questions 1–3. Use your time wisely; the questions are not of equal value. You must show all necessary working to receive full points for a problem. If you are unable to do one part of a problem, you may still use the result of that part in your answer to later parts of the exam. Calculators may be used if desired. **1.** *(30 points)*

(a) Let $L : K$ be a field extension. Explain carefully what is meant by the degree $[L : K]$ of the extension. If M is an intermediate field, what is the relation between $[L : M]$, $[M : K]$ and $[L : K]$?

(b) Explain what is meant by saying that a field extension $L : K$ is (i) simple, (ii) finite or (iii) algebraic.

(c) Show that $\mathbf{Q}(\sqrt{2}, \sqrt{7})$ is a simple extension of $\mathbf{Q}$.

(d) Let $F = K(t)$ be the field of rational functions in an indeterminate t over a field K, and $\alpha = t^3 + (2+t)^{-1} \in F$. Set $L = K(\alpha)$ (a subfield of F). Show that t is algebraic over L.

2. *(25 points)*

(a) Let K be a field, and $f \in K[t]$ be an irreducible polynomial over K. Explain how to construct a simple extension $L = K(\alpha)$ of K in which f has a zero α. How is irreducibility used in the proof that your L is a field? What is the relation between the degree of the polynomial f and the extension degree $[L : K]$?

(b) Taking $K = \mathbf{Z}_3$ in (a), find a suitable polynomial f so the resulting field L contains 9 elements. Express each of α^3 and $(\alpha+1)^{-1}$ in the form $c_0 + c_1\alpha$ with the $c_i \in \mathbf{Z}_3$. Are these expressions unique?

3. *(45 points)* Let $\alpha = \sqrt[5]{2} \in \mathbf{R}$ and $\omega = e^{2\pi i/5} \in \mathbf{C}$. Define polynomials f and g in $\mathbf{Q}[X]$ by $g = X^4 + X^3 + X^2 + X + 1$ and $f = X^5 - 2$.

(a) State Eisenstein's criterion and use it show $g(X+1)$ (and hence g) is irreducible over $\mathbf{Q}$. Prove that that $g = \min(\omega, \mathbf{Q}, X)$ and conclude that $[\mathbf{Q}(\omega) : \mathbf{Q}] = 4$.

(b)Show similarly that $f = \min(\alpha, \mathbf{Q}, X)$ and compute $[\mathbf{Q}(\alpha) : \mathbf{Q}]$. Conclude that $[\mathbf{Q}(\alpha, \omega) : \mathbf{Q}(\omega)] \leq 5$. (Hint; $\min(\alpha, \mathbf{Q}(\omega), X)$ divides f in $\mathbf{Q}(\omega)[X]$).

(c) Prove that $[\mathbf{Q}(\alpha, \omega) : \mathbf{Q}]$=20. (Hint; show the degree is divisible by 4 and by 5, and is less than or equal to 20.)

(d) Explain what is meant by an isomorphism from a field extension $K : M$ to a field extension $L : N$. Show that any automorphism of $\mathbf{Q}(\omega) : \mathbf{Q}$ (i.e. an isomorphism of this extension with itself) must map ω to a root ω^j, $1 \leq j \leq 4$ of g in $\mathbf{Q}(\omega)$.

(e) Suppose that $K(\beta) : K$ and $L(\gamma) : L$ are simple algebraic extensions. Give sufficient conditions on the minimum polynomials of α and β under which an isomorphism of fields $\theta : K \to L$ can be extended to an isomorphism of the two extensions. Use your condition to show that for any integer j with $1 \leq j \leq 4$ there is an automorphism of $\mathbf{Q}(\omega) : \mathbf{Q}$ mapping ω to ω^j.