Math 362 Exam 2; Fri April 4, 1997 12:55–1:45pm

Instructions. Answer all questions. You must show all necessary working to receive full points for a problem. Calculators may be used if desired. **1.** *(15 points)*

(a) Suppose that the point $(x, y) \in^2$ is constructible (by ruler and compass) from the points $(0, 0)$ and $(1, 0)$. What can one conclude about the field degrees $[(x):]$ and $[(y):]$?

(b) Use the result stated in your answer to (a) to give a careful proof that the point $(\sqrt[5]{2}, 0)$ is not constructible from $(0, 0)$ and $(1, 0)$.

2. *(35 points)*

(a) Explain what is meant by saying that a finite field extension L/K is (i) the splitting field of a polynomial $f \in K[t]$ (ii) a normal extension (iii) a separable extension.

(b) Under what conditions on the characteristic of the field F will any finite field extension E/F be separable.? Give an example of a field extension which is finite and normal but not separable.

(c) State the theorem giving the relationship between finite normal extensions and splitting fields. Use this result to prove that if L/K is a finite normal extension and M is an intermediate field, then L/M is normal.

(d) Let $L = (\sqrt[3]{5}, e^{2\pi i/3})$, $M = (\sqrt[3]{5})$ and $K =$. Which of the extensions L/K, L/M and M/K are (i) normal (ii) separable?

3. *(25 points)* Let L/K be a finite field extension.

(a) Give a careful definition of the Galois group $G = \Gamma(L/K)$.

(b) For an intermediate field M, how is the corresponding subgroup M^* of G defined? Also define the intermediate field $H^\dagger$ corresponding to a subgroup H of G.

(c) Prove that if $\sigma \in G$, then $(\sigma H \sigma^{-1})^\dagger = \sigma(H^\dagger)$, where $\sigma H \sigma^{-1}$ is the subgroup of G conjugate to H by σ and $\sigma(H^\dagger)$ denotes the image of $H^\dagger$ under σ.

4. *(25 points)* You may assume in your answer to this question that the polynomial $f = t^6 + t^5 + t^4 + t^3 + t^2 + t + 1 \in [t]$ is irreducible.

(a) Let $\omega = e^{2\pi i/7}$. Explain why f is the minimum polynomial of ω^i over for $i = 1, \ldots, 6$. Conclude that $L := (\omega)$ is a splitting field of f over and that for each $i = 1, 2, \ldots, 6$ there is a unique -automorphism σ_i of L satisfying $\sigma_i(\omega) = \omega^i$.

(b) Show that $\sigma_i \sigma_j = \sigma_k$ where $0 < k < 7$ is the remainder on division of ij by 7. Conclude that the Galois goup $\Gamma(L/)$ is a cyclic group of order 6 generated by σ_3.

(c) Let H be the subgroup of G generated by σ_2. Find the degree of the extension $L/H^{\dagger}$.