Final Exam for Math 362, Honors Algebra IV
May 4, 1998, 4:15–6:15pm.

1. (10 pts) Write down an irreducible polynomial which is not normal. Explain why your example has the stated properties.

   (10 pts) Is $[\mathbb{C} : \mathbb{R}]$ a Galois extension? Give a detailed answer.

2. (10 pts) Let

$$H := \mathrm{rowsp}_{\mathbb{Z}} \begin{pmatrix} 2 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 10 \end{pmatrix}.$$

Compute the invariant factor decomposition of the abelian group $\mathbb{Z}^3/H$.

(10 pts) Count the number of non-isomorphic abelian groups of order 720.

3. (10 pts) Show that $x^3 - t \in \mathbb{Z}_3(t)[x]$ is an irreducible and inseparable polynomial.

(10 pts) Write down a nontrivial field automorphism $GF(1024) \longrightarrow GF(1024)$.

4. (10 pts) Find the number of all $[5, 3]$ linear codes defined over $\mathbb{Z}_2$. Alternatively count the number of subspaces $V \subset (\mathbb{Z}_2)^5$ with $\dim V = 3$.

(10 pts) Give an example of a group of order 60 which is not solvable (=soluble).

5. (20 pts) Prove in detail that $GF(2^{30}) \supset GF(2)$ is a simple extension.

6. (20 pts) Prove that $\mathbb{Q}(\sqrt{5}+\sqrt{7}) \supset \mathbb{Q}$ is a Galois extension. Then compute the Galois group $\Gamma[\mathbb{Q}(\sqrt{5}+\sqrt{7}) : \mathbb{Q}]$ and work out the Galois correspondence indicating on one side the subgroups of Γ and on the other side the subfields of $\mathbb{Q}(\sqrt{5}+\sqrt{7})$.

7. a) (10 pts) Describe in detail the splitting field $\mathbb{F} \subset \mathbb{C}$ of the polynomial $x^8 - 1 \in \mathbb{Q}[x]$.

b) (10pts) Compute the Galois group $\Gamma[\mathbb{F} : \mathbb{Q}]$. What is the degree $[\mathbb{F} : \mathbb{Q}]$?

c) (10pts) Work out the Galois correspondence indicating on one side the subgroups of $\Gamma[\mathbb{F} : \mathbb{Q}]$ and on the other side the subfields of $\mathbb{F}$.