Math 362 Exam 1; Thursday Feb 25,1999 12:30–1:45pm

Instructions Answer questions 1–3. Use your time wisely; the questions are not of equal value. You must show all necessary working to receive full points for a problem. If you are unable to do one part of a problem, you may still use the result of that part in your answer to other parts of the exam. **1.** *(25 points)*

(a) Let $L : K$ be a field extension. Explain what is meant by the degree $[L : K]$ of the extension. If M is an intermediate field, what is the relationship between $[L : M]$, $[M : K]$ and $[L : K]$?

(b) Decide which of the field extensions (i) $\mathbf{C} : \mathbf{R}$ and (ii) $\mathbf{R}\colon \mathbf{Q}$ are (I) simple, (II) finite or (III) algebraic.

2. *(25 points)*(a) Let K be a field, and $f \in K[t]$ be an irreducible polynomial over K. Explain how to construct a simple extension $L = K(\alpha)$ of K in which f has a zero α. What is the relationship between the degree of the polynomial f and the degree of the extension $[L : K]$?

(b) Take $K = \mathbf{Z}_2$ and $f = t^3 + t + 1$ in (a). Explain why the elements $1, \alpha, \alpha^2$ form a basis of L over K. Express each of α^5 and $(\alpha + 1)^{-1}$ as a K-linear combination of these basis elements.

3. *(50 points)* Consider the field extension $\mathbf{Q}(\alpha, \omega) : \mathbf{Q}$ where $\alpha = \sqrt[3]{2}$ and $\omega = e^{2\pi i/3}$.

(a) Show that $f = \min(\omega, \mathbf{Q}, X) = X^2 + X + 1$ and $g = \min(\alpha, \mathbf{Q}, X) = X^3 - 2$. Show that the roots of f in the complex numbers are ω^j, $1 \le j \le 2$ and those of g are $\alpha\omega^i$, $0 \le i \le 2$.

(b) Show that $[\mathbf{Q}(\alpha, \omega) : \mathbf{Q}] = 6$ (Hint: use (a) to show that the degree is divisible by 2 and by 3 and is less than or equal to 6).

(c) Suppose that $K(\beta) : K$ and $L(\gamma) : L$ are simple algebraic extensions, and $\theta\colon K \to L$ is a field isomorphism. Give the necessary and sufficient condition relating θ and the minimum polynomials of β and γ so that there is an isomorphism of the extensions $K(\beta) : K$ with $L(\gamma) : L$ compatible with θ and mapping β to γ.

(d) Use (c) to show that the Galois group of $\mathbf{Q}(\alpha, \omega) : \mathbf{Q}$ consists of six elements $\sigma_{i,j}$ ($0 \le i \le 2$, $1 \le j \le 2$) with $\sigma_{i,j}(\alpha) = \alpha\omega^i$ and $\sigma_{i,j}(\omega) = \omega^j$.

(e) Show the Galois group in (d) is not abelian. Can you identify which familiar group with 6 elements is isomorphic to the Galois group?