Math 362 Exam 2: Tuesday Mar 30,1999; 12:30–1:45pm

Instructions Answer questions 1–2. You must show all necessary working to receive full points for a problem. If you are unable to do one part of a problem, you may still use the result of that part in your answer to other parts of the exam. **1.** *(50 points)* Let $L : K$ be a finite field extension.

(a) Give a careful definition of the Galois group $G = \Gamma(L : K)$. (b) For an intermediate field M, how is the corresponding subgroup M^* of the Galois group defined? Also define the intermediate field $H^\dagger$ corresponding to a subgroup H of G. Prove carefully that $H \subseteq (H^\dagger)^*$.(c) Explain carefully what is meant by saying that the extension $L : K$ is normal and separable. Give an example of a finite extension which is not normal.(d) Under what assumptions on the extension $L : K$ will the Galois correspondence be bijective. Assuming that it is bijective, show that if M is an intermediate field and $\tau \in G$ then $(\tau(M))^* = \tau M^* \tau^{-1}$. (e) Assume the Galois correspondence is bijective. Show that if M and N are intermediate fields such that $M \cap N = K$, then $M^* \cup N^*$ generates the group G.

2. *(50 points)* Let ω denote the complex number $\omega = e^{\pi i/4} = \frac{1}{\sqrt{2}}(1 + i)$ where $i^2 = -1$.(a) Show that $\mathrm{Min}(\omega, \mathbf{Q}, X) = f(X)$ where $f(X) = X^4 + 1$. (Hint: show f is irreducible over $\mathbf{Q}$ by considering $f(X + 1)$). What is the degree $[\mathbf{Q}(\omega) : \mathbf{Q}]$? (b) Show that the roots of f in $\mathbf{C}$ are ω^j for $j = 1, 3, 5, 7$. Conclude that $\mathbf{Q}(\omega)$ is a splitting field for f over $\mathbf{Q}$. Is $\mathbf{Q}(\omega) : \mathbf{Q}$ a normal extension? Is it a separable extension? Is the Galois correspondence bijective for this extension? (c) Let $G = \Gamma(\mathbf{Q}(\omega) : \mathbf{Q})$ be the Galois group. Show that G has four elements and conclude $G = \{\tau_1, \tau_3, \tau_5, \tau_7\}$ where $\tau_j(\omega) = \omega^j$ for each $j = 1, 3, 5, 7$.

(d) Show that $\tau_j^2 = \tau_1$ for all j and conclude that G is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$.

(e) Draw diagrams of all the subgroups of G and the corresponding intermediate fields of $\mathbf{Q}(\omega) : \mathbf{Q}$ illustrating the Galois correspondence. (As a check on your answer, you should find five subgroups including G and the trivial subgroup.)