Math 362 Final Exam: Wednesday May 5,1999; 2:00–4:00pm

Instructions Answer questions 1–4. You must show all necessary working to receive full points for a problem. If you are unable to do one part of a problem, you may still use the result of that part in your answer to other parts of the exam. **1.** *(30 points)* Let $L : K$ be a field extension.

(a) Explain carefully what is meant by saying that this extension is (i) finite (ii) algebraic (iii) simple (iv) normal or (v) separable. (b) For each of these properties, give an example of a field extension $L : K$ with $L \neq K$ which has this property, and an example of a field extension which doesn't have this property (in this part of the exam only, you are NOT required to prove that the extensions you give do or do not have the property).

2. *(40 points)* Let K be a field of characteristic zero. (a) Explain what is meant by a solvable group. Give an example of a solvable group of order 240 and an example of a group of order 240 which is not solvable. (b) Explain what is meant by saying that an extension $L : K$ of K is radical. (c) Define the Galois group of a polynomial $f \in K[t]$. Explain what is meant by saying that f is solvable by radicals. (d) Carefully state the theorem giving necessary and sufficient conditions for f to be solvable by radicals. (e) State conditions on a polynomial f of prime degree p over the rational numbers which are sufficient to ensure it has the symmetric group S_p as its Galois group. (f) Use the theorems in parts (d) and (e) to show that the polynomial $t^5 - 4t - 2$ over $\mathbf{Q}$ is not solvable by radicals.

3. *(40 points)* Let ω denote the complex number $\omega = e^{2\pi i/10}$, so $\omega^{10} = 1$.(a) Let $f(X) = X^4 - X^3 + X^2 - X + 1$. Show that $f(X)(X+1)(X^5-1) = X^{10} - 1$. Conclude that the four roots of f in the complex numbers are ω^i for $i = 1, 3, 7, 9$.(b) Show that $\mathrm{Min}(\omega, \mathbf{Q}, X) = f(X)$. (Hint: show f is irreducible over $\mathbf{Q}$ by considering $f(X-1)$). (c) State necessary and sufficient conditions on a field extension $L : K$ of finite degree for the Galois correspondence to be bijective. Prove that these conditions are satisfied by the extension $\mathbf{Q}(\omega) : \mathbf{Q}$.

(d) Let $G = \Gamma(\mathbf{Q}(\omega) : \mathbf{Q})$ be the Galois group. Show that $G = \{\tau_1, \tau_3, \tau_7, \tau_9\}$ where $\tau_j(\omega) = \omega^j$ for $j = 1, 3, 7, 9$, and that $\tau_i\tau_j = \tau_k$ where $k \equiv ij \pmod{10}$. Conclude that G is a cyclic group.

(e) Draw diagrams of all the subgroups of G and the corresponding intermediate fields of $\mathbf{Q}(\omega) : \mathbf{Q}$ illustrating the Galois correspondence.

4. *(40 points)* Let $K = \mathbf{Z}_2$ and $f = t^4 + t + 1 \in K[t]$. Let M be a splitting field of f over K and $L = K(\alpha)$ where α is a root of f in M.

(a) Show that f is irreducible over k and conclude that $1, \alpha, \alpha^2, \alpha^3$ is a K-basis of L. How many elements does L have? For which integers q does L have a subfield with q elements? (b) Explain why $M = L$. (c) Show that the element $u = \alpha + 1$ is a generator of the multiplicative group of L. (d) Let $\theta : L \to L$ be the Frobenius automorphism of L, given by $\theta(x) = x^2$ for $x \in L$. Show that the fixed field of θ^2 is $\{0, 1, u^5, u^{10}\}$. What is $\mathrm{Min}(u^{10}, K, X)$?

(e) For a positive integer n, describe the factorization into irreducible polynomials over K of the polynomial $t^{2^n} - t$. Determine the number of irreducible polynomials of degree 12 over K.