MATHEMATICS 362 **Name** ____________________

Test 3

April 9, 2001 3 questions

1. (*30 points*)

Let $L : K$ be a field extension which is finite, normal and separable.

(a) By considering the non-identity elements, show that a group of finite order n has at most $\binom{n-1}{m-1}$ subgroups of order m.

(b) Prove that there are only finitely many intermediate fields M in the extension $L : K$. [*Name any theorem you use in your argument.*]

(c) If $[L : K] = 4$, find the best upper bound you can for the number of intermediate fields in the extension $L : K$. [*Name any theorem you use in your argument.*]

2. (*30 points*)

Let $L : K$ be a field extension which is finite, normal and separable. Let G denote the Galois group $\Gamma(L : K)$. Suppose that M is an intermediate field, and M^* is the corresponding subgroup of G.

(a) Let $\phi \in G$, so that $\phi(M)$ is another intermediate field, which may or may not be equal to M. (It would be equal to M if $M : K$ is normal.) Prove that $\phi(M)^* = \phi M^* \phi^{-1}$.

(b) A theorem proved in class showed that the following two statements are equivalent.

(i) $M : K$ is a normal extension.
(ii) M^* is a normal subgroup of G.

Give the proof of the part of the theorem that says that (i) implies (ii).

(c) If $M : K$ is a normal extension, how are M^*, G and $\Gamma(M : K)$ related? [*No proof needed.*]

3. (*40 points*)

Let L be the splitting field of $t^n - 1$ over a field K, and let U be the set of all roots of $t^n - 1$ in L.

(a) Show that U is a subgroup of the multiplicative group of the nonzero elements of L.

(b) State carefully a theorem of finite group theory which can be used to show that U is cyclic. [*No proof needed.*]

(c) Give the proof that the Galois group $\Gamma(L : K)$ is abelian.

(d) What can be said about the Galois group G of a polynomial of form $t^n - b$ over a field of characteristic 0? [*No proof needed.*]