MATHEMATICS 362 **Name** ______________________

Final examination

May 7, 2001 6 questions

In questions 2 and 3, $\mathbb{Q}$ denotes the field of rational numbers.

1. (*30 points*)

[*No proofs are required in this question.*]

Let K, L, M be fields such that $K \subseteq L \subseteq M$. Suppose α, β are elements such that $L = K(\alpha)$, $M = L(\beta)$. Also suppose $[L : K] = m$, $[M : L] = n$.

(a) Explain what is meant by the statement $L = K(\alpha)$.

(b) Explain what is meant by the equation $[L : K] = m$.

(c) Define the minimal polynomial of α over K, and give its relation to $[L : K]$.

(d) Give a basis of L over K.

(e) Give a basis of M over K.

2. (*28 points*)

(a) State carefully a theorem connecting derivatives with multiple roots of polynomials. [*No proof needed.*]

(b) Which of the following polynomials in $\mathbb{Q}[t]$ have a multiple root (in some extension field of $\mathbb{Q}$)? Find the multiple roots in the cases where they exist.

(i) $t^3 - 3t - 2$.

(ii) $t^6 - 3t^2 - 2$

(iii) $t^{17} + 6t^9 + 200t^3 + 18t - 10$

(*Give reasons for your answers.*)

3. (*12 points*)

(a) Define the notion of a splitting field for a polynomial $f(t)$ over a field K.

(b) Give a polynomial whose splitting field over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. [*No proof needed.*]

4. (*16 points*)

[*In this question, "constructible" means "constructible by straightedge and compass, starting with the points (0,0) and (1,0)"*]

Some of the points on the cubic curve $y = x^3$ are constructible, e.g. $(1, 1)$, but some are not, e.g. $(\sqrt[3]{2}, 2)$. Determine whether or not the intersections of the curve with the ellipse $2x^2 + y^2 = 6$ are constructible.

5. (*32 points*)

[*No proofs are required in this question.*]

Let K be a field of characteristic 0.

(a) Explain what is meant by saying an extension field L of K is a radical extension of K.

(b) Define the Galois group of a polynomial f in $K[t]$.

(c) Explain what is meant by saying that a polynomial f in $K[t]$ is solvable by radicals.

(d) Explain what is meant by saying that a group G is solvable.

(e) Carefully state the theorem giving the relationship between solvable groups and solvability of polynomials by radicals.

6. (*32 points*)

Let $L : K$ be a field extension, with Galois group $G = \Gamma(L : K)$.

(a) If M is an intermediate field, define the subgroup M^* of G corresponding to M in the Galois correspondence. [*You don't have to show it's actually a subgroup.*]

(b) If M_1 and M_2 are intermediate fields, and $M_1 \subseteq M_2$, state and prove the relationship between M_1^* and M_2^*.

(c) If M is an intermediate field, and N is the fixed field of M^* in L, show that $M \subseteq N$.

(d) Give a condition on the extension $L : K$ under which $M = N$ holds in (c), for every intermediate field M. [*No proof needed.*]

(e) If $\alpha_1, \alpha_2 \in L$, $M_1 = K(\alpha_1)$, $M_2 = K(\alpha_2)$, $M = K(\alpha_1, \alpha_2)$, show that $M^* = M_1^* \cap M_2^*$.