MATHEMATICS 362 **Name** ______________________

Test 1

February 7, 2001 4 questions

In questions 2, 3, 4, $\mathbb{Q}$ denotes the field of rational numbers.

1. (*25 points*)

(a) Define what it means for a field extension $L : K$ to be a finite extension, and define the degree $[L : K]$.

(b) Let K, L, M be fields, with $K \subseteq L \subseteq M$, and suppose $M : L$ and $L : K$ are finite extensions. Explain carefully how to obtain a basis B of M over K, from a basis of M over L and a basis of L over K. (*Proof that B is actually a basis is not required.*)

(c) Suppose K, L, M, N are fields, with $K \subseteq L \subseteq M \subseteq N$, and suppose that $[M : K] = 15$, $[N : L] = 6$. Show that $[N : K]$ must be either 30 or 90 .

2. (*25 points*) Let $L : K$ be a field extension, $\alpha \in L$.

(a) Explain what is meant by $K(\alpha)$.

(b) If f is an irreducible polynomial of degree n in $K[t]$, and α is a root of f, write down a basis for $K(\alpha)$ over K. (*Proof not needed.*)

(c) Show that the polynomial $g = t^2 + t + 1$ is irreducible over the field $\mathbb{Q}$.

(d) Suppose the complex number α is a root of the polynomial g of (c) above. If $a, b, c, d \in \mathbb{Q}$, express the product $(a + b\alpha)(c + d\alpha)$ in the form $e + f\alpha$, where $e, f \in \mathbb{Q}$.

3. (*25 points*)

(a) If α, β are both algebraic over a field K, give a condition involving polynomials under which there exists a K-isomorphism (i.e., an isomorphism leaving every element of K fixed)

$$\phi : K(\alpha) \to K(\beta),$$

such that $\phi(\alpha) = \beta$. (*No justification needed.*)

For the next two parts of the question, let α be as in Question 2(d), i.e., α is a complex root of the polynomial $g = t^2 + t + 1$.

(b) Show that there exists a $\mathbb{Q}$-automorphism

$$\phi : \mathbb{Q}(\alpha) \to \mathbb{Q}(\alpha),$$

such that $\phi(\alpha) = -1 - \alpha$.

(c) If $\gamma = a + b\alpha$, where $a, b \in \mathbb{Q}$, compute the product $\gamma \cdot \phi(\gamma)$, and find a formula for $(a + b\alpha)^{-1}$.

4. (*25 points*)

(a) Give a careful statement (*without proof*) of a theorem connecting field extensions with geometric constructibility.

(b) It is easy to bisect angles by means of straightedge and compass. Given a number θ, let $\alpha = \cos\theta, \beta = \cos 2\theta$. By using a suitable trigonometric identity, show that

$$\mathbb{Q}(\beta) \subseteq \mathbb{Q}(\alpha),$$
$$[\mathbb{Q}(\alpha) : \mathbb{Q}(\beta)] \leq 2.$$

(c) If $\alpha_n = \cos(\pi/2^n)$, where n is a positive integer, show that

$$[\mathbb{Q}(\alpha_n) : \mathbb{Q}] = 2^m,$$

where $m < n$.