Math 365, Test # 1 / 100

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Name:________________________________

For a proof (Problems 3–7), please quote the theorem you use for each step.

1. (32 points). True or False. (No proof or counter example needed).

(a). The countable union of countable sets is a countable set.

(b). The complex field is not an ordered field.

(c). If $\{G_n\}$ is open, then $\bigcap_{n=1}^{\infty} G_n$ is open.

(d). If E is a nonempty bounded set in Q (the rationals), then $\sup E$ exists and is also a rational.

(e). If G is open, then G is not closed.

(f). The product of two irrationals is always irrational.

(g). If a is rational, b is irrational, then $a + 9b$ is always irrational.

(h). Let X be a metric space with a distance function $d(p, q)$, then any neighborhood must contain infinitely many points.

2. (20 points).

(a). State PRECISELY the definition of a compact set.

(b). State PRECISELY the definition of i) a metric, and ii) a metric space.

(c). State PRECISELY the definition for a limit point for a set E.

(d). State PRECISELY the definition for a perfect set.

3. (10 points). If x and y are complex numbers, prove that $\Big||x| - |y|\Big| \le |x - y|$.

4. (10 points).

(i). Do E and $\bar{E}$ always have the same interiors? Prove or give a counter example.

(ii). Do E and E° always have the same closure? Prove or give a counter example.

5. (10 points). Suppose that X is a metric space with d. Suppose that $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} x_n = p.$$

(We say $\lim_{n\to\infty} x_n = p$ if for any neighborhood N of p, there is a $N > 0$ such that $x_n \in N$ for all $n > N$.)

Let

$$E = \{x_n\}_{n=1}^{\infty} \cup \{p\}.$$

(i) Is E a closed set? Prove or give a counter example.

(ii) Is E a compact set? Prove or give a counter example.

6. (8 points). Please explain:

(a) We know that the interval $(0, 1)$ of the real line is not a compact set. However, here is a "proof".

Proof: We know that $[0, 1]$ is a closed, compact set. $(0, 1)$ is a subset of $[0, 1]$. Therefore by Theorem, the set $(0, 1)$, being a subset of a compact set, is also a compact set.

What is wrong with this proof?

(b) We know that the set of real numbers in (0,1) is not countable. However, here is a "proof".

Proof: We write each number as $0.q_1q_2q_3\cdots$. Each q_j takes the value $0, 1, 2, 3, 4, 5, 6, 7, 8$ or 9. Thus q_j has 10 possible values. Now if we let q_j to be all these possible values, all real numbers in (0,1) are taken care of. There are only countably number of digits. Therefore, the set (0,1), as a countable union of finite sets, is countable.

What is wrong with this proof?

7. (10 points). Let $J = \{1, 2, 3, \cdots\}$. Let $\mathcal{E}$ be the set of all subsets of J. Prove that $\mathcal{E}$ is uncountable.

(The elements of $\mathcal{E}$ are subsets of J, such as: $\{1, 3, 8, 9, 99\}$, $\{1, 3, 5, 7, 9, \cdots\}$, $\cdots$)