Math 365, Test # 3   / 100

Dec 6, 1996
Instructor: Bei Hu

Name:______________________________

1. (25 points). True or False. (No proof or counter example needed).

(a). A continuous function defined on a bounded and closed set of R^k is always uniformly continuous.

(b). Suppose $a_n > 0$. If $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, then the series $\sum a_n$ is divergent.

(c). In metric spaces, if the inverse function of a continuous function exists, then it must be continuous.

(d). A convergent series is always absolutely convergent.

(e). If $f : X \longrightarrow Y$ is a continuous function and $E \subset Y$ is a connected set, then $f^{-1}(E)$ is connected.

2. (20 points).

(a). State PRECISELY the definition for the radius of convergence for a power series.

(b). State PRECISELY the formula of Summation by parts.

(c). State PRECISELY the intermediate value theorem.

(d). State PRECISELY two equivalent definitions for a continuous function.

3. (10 points). Determine the radius of convergence for the following power series.

(a). $\sum_{n=0}^{\infty}(2+n^3)z^n$.

(b). $\sum_{n=0}^{\infty}\frac{4^n}{n^7}z^n$.

4. (10 points). Determine the convergence for the following series, indicating the theorem that you used.

(a). $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

(b). $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$.

5. (8 points) Let f be a real function on a metric space X. Prove that the zero set of f, defined by $Z(f) = \{p \in X \,\big|\, f(p) = 0\}$, is a closed set in X.

6. (7 points) Suppose that $a_n \geq 0$ and $\sum a_n$ converges. Prove that $\frac{\sqrt{a_n}}{n}$ converges.

7. (10 points) Let $f(x) = \sin\left(\frac{1}{x}\right)$ and $g(x) = xf(x)$.

(a). Is $f(x)$ continuous on $(0, 1)$?

(b). Is $g(x)$ continuous on $(0, 1)$?

(c). Is $f(x)$ uniformly continuous on $(0, 1)$?

(d). Is $g(x)$ uniformly continuous on $(0, 1)$?

Justify your answers.

8. (10 points). Let f be a continuous function from the metric space X to the metric space Y.

(a). Prove that for $E \subset X$, $f(\bar{E}) \subset \overline{f(E)}$.

(b). Is the equality $f(\bar{E}) = \overline{f(E)}$ valid? (Prove or give a counter example).

(c). Is the equality in (b) valid if $X = (-\infty, \infty)$ and E is bounded? (Prove or give a counter example).