Math 365, Final Test /150

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Name:______________________________

1. (35 points). True or False. (No proof or counter example needed).

(a). A finite set in a metric space is compact.

(b). If $E \subset X$ is a connected set and f is a continuous function on X, then $f(E)$ is also a connected set.

(c). If a function $f(x)$ is differentiable at a point $x = a$, then $f(x)$ is also continuous at the point $x = a$.

(d). A nonempty perfect set in a metric space is uncountable.

(e). If a sequence converges, then any rearrangement of the series also converges.

(f). In a metric space X, let $\{F_n\}$ be a sequence of nonempty compact sets such that $F_n \supset F_{n+1}$. Then the intersection $\bigcap_{n=1}^{\infty} F_n$ is nonempty.

(g). In a metric space, a continuous function maps a compact set to a compact set.

2. (25 points).

(a). State PRECISELY the Lagrange mean value theorem.

(b). State PRECISELY the Taylor's formula (Assumptions and conclusions).

(c). State PRECISELY Weierstrass theorem.

(d). State PRECISELY the root test for a series.

(e). State PRECISELY the least upper bound principle.

3. (10 points). Determine the radius of convergence for the following power series.

(a). $\sum_{n=0}^{\infty} \frac{1}{(2n)!} z^n$.

(b). $\sum_{n=0}^{\infty} \frac{3^n}{(1+4n^3+6n^5+100n^{99})} z^n$.

4. (10 points). Determine the convergence for the following series, indicating the theorem that you used.

(a). $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin\left(\frac{\pi}{4} n\right)$.

(b). $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$.

5. (10 points) Suppose that $C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$, where $C_0, \cdots, C_n$ are real constants. Prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

6. (10 points) f is a real valued function defined on R^1. Suppose that there exists a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t.

Define $x_0 = 1$ and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, \cdots$. Prove that

(a) the limit $x = \lim_{n\to\infty} x_n$ exists,

(b) the x obtained in (a) satisfies $f(x) = x$.

7. (10 points) Find the limits. Quote all theorems that you used.

(a). $\lim_{x\to 0} \frac{\cos(x^{49})-1}{x^{98}}$

(b). $\lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}}$

(Hint: is there a theorem about the relationship between root and ratio?)

8. (10 points). Suppose that $\lim_{n\to\infty} a_n = a$ and let

$$\sigma_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Prove that $\lim_{n\to\infty} \sigma_n = a$.

9. (10 points). Suppose that the function $f : X \to R^1$ is continuous. The level set at level β (β is a real number) is defined as $L_\beta = \left\{ p \in X \middle| f(p) = \beta \right\}$.

Prove that for any real β, the level set L_β is a closed set.

10. (10 points).

(a) Please explain: We know that the set $[0, \infty)$ is not compact. However, here is a "proof".

Proof: Define the following metric $d(x, y) = \frac{|x-y|}{1+|x-y|}$. (This is a metric, and you don't have to verify this.) With this metric, the set $[0, \infty)$ is a closed and bounded set in the Euclidean space R^1. Therefore, by Heine-Borel theorem, $[0, \infty)$ is a compact set.

What's wrong with this proof?

(b) Suppose that $f : [a, b] \to R^2$ is a vector valued function which is differentiable everywhere on $[a, b]$. Is the mean value theorem valid for f? namely, there exists $\xi \in (a, b)$ such that $f(a) - f(b) = (b - a)f'(\xi)$.

Prove if it is true. Give a counter example if it is false.

11. (10 points). Is the following statement true? Prove if it is true, give a counter example if it is false.

Suppose that $\{x_n\}$ is a Cauchy sequence on the metric space X and f is a continuous function from X to Y. Then the sequence $\{f(x_n)\}$ is a Cauchy sequence on Y.