Name: ________________________________ **Instructor:** Jeffrey Diller

Math 365: Honors Real Analysis I
Fall Semester 2003
Exam 1
Monday, October 6

This Examination contains 4 problems. Counting the front cover and blank pages, the exam consists of 8 sheets of paper.

Scores

Question	Possible	Actual
1	25	
2	25	
3	30	
4	20	
Total	100	

GOOD LUCK

1. Do each of the following (4 points each).

(a) Define *cauchy sequence.*

(b) Define *convergent series.*

(c) State the root test.

(d) Define *compact set.*

(e) Define *upper limit of a sequence.*

2. Five of the following ten assertions are false. Identify them and give counterexamples on the following page. Note that you do not have to justify your counterexample. (5 points each)

(a) A convergent sequence is bounded.

(b) A Cauchy sequence converges.

(c) If $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

(d) Every sequence in a compact set has a convergent subsequence.

(e) If A is a set and $B \subset A$ is a proper subset (i.e. $B \neq A$), then the cardinalities of A and B are not equal.

(f) If U is an open set, then the interior of the closure of U is equal to U.

(g) If $\sum_{n=0}^{\infty} a_n$ converges, and $\{a_{n_j}\}_{j\in}$ is a subsequence, then $\sum_{j=0}^{\infty} a_{n_j}$ converges.

(h) Let $\{a_n\}$ be a sequence in a metric space X. Let E be the set of all limits of subsequences of $\{a_n\}_{n\in}$. Then E is closed.

(i) Let $\{a_n\}_{n\in} \subset$ be a sequence such that $\lim_{n\to\infty} |a_{n+1} - a_n|$ converges. Then $\{a_n\}$ converges.

(j) The product of two irrational numbers is irrational.

3. Do three of the following four problems. If you turn in solutions to all four, I will simply grade the first three. (10 points each)

(a) Let $\{a_n\}_{n\in} \subset$ be a sequence such that $\sum |a_n|$ converges. Show that $\sum a_n$ converges.

(b) Let X be a metric space and $E \subset X$. Prove that the interior of E is open.

(c) Prove that there is no $x \in$ such that $x^3 = 24$.

(d) Prove that a closed subset of a compact set is compact.

4. Take home problem(s), due Monday 10/14 by class time. Do one of two—note that the second problem is both longer and worth more points than the first one. If you turn in solutions to both, I will grade only the first. The only resources you are to use in solving these problems are your textbook and yourself. I will answer questions about the problems only insofar as they clarify what is written here.

(a) Let $\{a_n\} \subset X$ be a sequence in a complete metric space such that $\sum_{n=0}^{\infty} d(a_{n+1}, a_n)$ converges. Show that $\{a_n\}$ converges. (15 points)

(b) Let $U \subset$ be an open set. Complete the following outline to show that U is a finite or countable union of mutually disjoint intervals. (20 points total)

- For $x, y \in U$, let us say that $x \sim y$ if and only if there is an open interval $(a, b) \subset U$ containing both x and y. Show that $\sim$ is an equivalence relation.
- For each $x \in U$, let $U(x)$ be the equivalence class of x. Show that $U(x)$ is an open interval.
- Show that there are at most countably many distinct equivalence classes $U(x) \subset U$.
- Conclude. Then draw a little filled in square at the end of your proof, taking scrupulous care to ensure that all four sides have the same length and that all four angles measure $\pi/2$.