1. Do each of the following (4 points each).

(a) Define *cauchy sequence.*

A sequence $\{x_n\}_{n\in}$ in a metric space X is said to be Cauchy if for every $\epsilon > 0$, there exists $N \in$ such that $d(x_n, x_m) < \epsilon$ whenever $n, m \geq N$.

(b) Define *convergent series.*

A series $\sum_{n=0}^{\infty} a_n$ is said to be convergent if the corresponding sequence $\{s_n\}_{n\in}$ of partial sums

$$s_n = a_1 + \ldots + a_n$$

converges.

(c) State the root test.

Let $\sum_{n=0}^{\infty} a_n$ be a series of non-negative real numbers. Let $L = \limsup |a_n|^{1/n}$. Then

- if $L > 1$, the series diverges;
- if $L < 1$, the series converges.

(d) Define *compact set.*

A set $E \subset X$ in a metric space X is said to be compact if for every open cover $\mathcal{U}$ of E, there exist finitely many elements $U_1, \ldots, U_n \in \mathcal{U}$ such that $E \subset U_1 \cup \ldots \cup U_n$.

(e) Define *upper limit of a sequence.*

Let $\{a_n\}$ be a sequence and E be the set of all limits of convergent subsequences of $\{a_n\}$. Then $\limsup a_n = \sup E$.

2. Five of the following ten assertions are false. Identify them and give counterexamples on the following page. Note that you do not have to justify your counterexample. (5 points each)

(a) A convergent sequence is bounded. *True*

(b) A Cauchy sequence converges. *False*

Let $X =$ and $\{a_n\} \subset X$ be a sequence of positive numbers with $a_n^2 \to 2$. Then $\{a_n\}$ is Cauchy but does not converge.

(c) If $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$. *True*

(d) Every sequence in a compact set has a convergent subsequence. *True*

(e) If A is a set and $B \subset A$ is a proper subset (i.e. $B \neq A$), then the cardinalities of A and B are not equal. *False*

Let $A =$ and $B =$, for instance.

(f) If U is an open set, then the interior of the closure of U is equal to U. *False*

Let $U = (0,1) \cup (1,2) \subset$.

(g) If $\sum_{n=0}^{\infty} a_n$ converges, and $\{a_{n_j}\}_{j\in}$ is a subsequence, then $\sum_{j=0}^{\infty} a_{n_j}$ converges. *True*

(h) Let $\{a_n\}$ be a sequence in a metric space X. Let E be the set of all limits of subsequences of $\{a_n\}_{n\in}$. Then E is closed. *True*

(i) Let $\{a_n\}_{n\in} \subset$ be a sequence such that $\lim_{n\to\infty} |a_{n+1} - a_n|$ converges. Then $\{a_n\}$ converges. *False*

Let $a_n = 1/1 + \ldots 1/n$.

(j) The product of two irrational numbers is irrational. *False*

For example, $\sqrt{2}\sqrt{2} = 2$.

3. Do three of the following four problems. If you turn in solutions to all four, I will simply grade the first three. (10 points each)

(a) Let $\{a_n\}_{n\in} \subset$ be a sequence such that $\sum |a_n|$ converges. Show that $\sum a_n$ converges.

Let $s_n = a_1 + \ldots + a_n$. Then it is enough to show that $\{s_n\}$ is Cauchy. On the other hand, if

$$t_n = |a_1| + \ldots + |a_n|,$$

then $\{t_n\}$ is Cauchy by hypothesis. So let $\epsilon > 0$ be given, and choose $N \in$ such that $|t_n - t_m| < \epsilon$ for all $n, m \geq$. Then if $m \geq n \geq$, we also have

$$|s_m - s_n| = |a_{n+1} + \ldots a_m| \leq |a_{n+1}| + \ldots |a_m| = |t_m - t_n| < \epsilon.$$

Hence $\{s_n\}$ is Cauchy, and we're on to the next problem.

(b) Let X be a metric space and $E \subset X$. Prove that the interior of E is open.

Let U be the interior of E. Given a point $p \in U$, we must show that p is an interior point of U. By definition of U, we know that there is $r > 0$ such that $N_r(p) \subset E$. But then if $q \in N_r(p)$, we also have

$$N_{r-d(p,q)}(q) \subset N_r(p) \subset E.$$

Hence $q \in U$, too. But $q \in N_r(p)$ was arbitrary, so $N_r(p) \subset U$. We conclude that p is an interior point of U, as desired.

(c) Prove that there is no $x \in$ such that $x^3 = 24$.

Suppose there is such an x. We can write $x = p/q$ where p and q have no common divisors. Then $p^3 = 24q^3$. In particular, all factors of 24 divide p^3. Therefore, the prime factors, such as 3 divide p itself. So we can write $p = 3k$ for some $k \in$. This gives us that $9k^3 = 8q^3$. Now we see that 3 divides q^3 and therefore also q. We conclude that 3 is a common factor of p and q, contradicting our initial assumption that p and q had no common divisors. This shows that x cannot exist.

(d) Prove that a closed subset of a compact set is compact.

Let E be a closed subset of a compact set K in a metric space X. Let $\mathcal{U}$ be an open cover of E. Then $\mathcal{U} \cup \{X - E\}$ is an open cover of K. So by hypothesis, we have elements $U_1, \ldots, U_n \subset \mathcal{U}$ such that

$$K \subset U_1 \cup \ldots \cup U_n \cup X - E.$$

It follows immediately that $X \subset U_1 \cup \ldots \cup U_n$. Therefore, X is compact, and this proof is closed. Get it, *closed*?

4. Take home problem(s), due Monday 10/14 by class time. Do one of two—note that the second problem is both longer and worth more points than the first one. If you turn in solutions to both, I will grade only the first. The only resources you are to use in solving these problems are your textbook and yourself. I will answer questions about the problems only insofar as they clarify what is written here.

(a) Let $\{a_n\} \subset X$ be a sequence in a complete metric space such that $\sum_{n=0}^{\infty} d(a_{n+1}, a_n)$ converges. Show that $\{a_n\}$ converges. (15 points)

Since X is complete it is enough to show that $\{a_n\}$ is Cauchy. Let $s_n = d(a_2, a_1) + \ldots d(a_{n+1}, a_n)$ be the partial sums of the series above. Then we know that $\{s_n\}$ is Cauchy by hypothesis. Given $\epsilon > 0$ choose $N \in$ such that

$$|s_n - s_m| < \epsilon$$

for all $n, m \geq N$. Then if $m \geq n \geq N + 1$ we have from the triangle inequality that

$$d(a_n, a_m) \leq d(a_{n+1}, a_n) + \ldots d(a_m, a_{m-1}) = |s_{n-1} - s_m| < \epsilon.$$

So $\{a_n\}$ is Cauchy, like we wanted it to be.

(b) Let $U \subset$ be an open set. Complete the following outline to show that U is a finite or countable union of mutually disjoint intervals. (20 points total)

- For $x, y \in U$, let us say that $x \sim y$ if and only if there is an open interval $(a, b) \subset U$ containing both x and y. Show that $\sim$ is an equivalence relation.

  Fix $x \in U$. Since U is open, we have $r > 0$ such that $N_r(x) = (x - r, x + r) \subset U$. Therefore $x \sim x$, and $\sim$ is reflexive. Moreover, if $x \sim y$, then there exists $(a, b) \subset U$ containing both x and y, so $y \sim x$, too. Therefore $\sim$ is symmetric. Finally, if $x \sim y$ and $y \sim z$, then we have open intervals $(a, b), (c, d) \subset U$ such that $x, y \in (a, b)$ and $y, z \in (c, d)$ In particular, $(a, b) \cap (c, d) \neq \emptyset$, so $(a, b) \cup (c, d)$ is again an open interval. Therefore, $x \sim z$, and $\sim$ is transitive.

- For each $x \in U$, let $U(x)$ be the equivalence class of x. Show that $U(x)$ is an open interval.

  Fix $x \in U$ and consider any point $y \in U(x)$. Then $x, y \in (a, b) \subset U$ for some (a, b). This implies that $(a, b) \subset U(x)$. So y is an interior point of $U(x)$, and $U(x)$ is open. Suppose now that $y, z \in U(x)$ and $y < t < z$. Then by transitivity, $y \sim z$. That is, there is an interval $(a, b) \in U$ such that $y, z \in (a, b)$. Again by transitivity, we see that $(a, b) \subset U(x)$. In particular $t \in U(x)$. So $U(x)$ is an interval.

- Show that there are at most countably many distinct equivalence classes $U(x) \subset U$.

  Fix $x \in X$. Since $U(x)$ is open, we can choose a point $q = q(x) \in \cap U(x)$. Now if $y \in X$ is another point, we have either $U(x) = U(y)$, in which case, we can take $q(y) = q(x)$; or $U(x) \cap U(y) = \emptyset$, in which case, we definitely ahve $q(x) \neq q(y)$.

So if we define $f(U(x)) = q(x)$, we see that f is an injective function from the set of equivalence classes of $\sim$ into . In particular, the cardinality of the set of equivalence classes is the same as that of some subset of , so it is therefore finite or countable.

- Conclude. Then draw a little filled in square at the end of your proof, taking scrupulous care to ensure that all four sides have the same length and that all four angles measure $\pi/2$.

  There's not really much left to say. The equivalence classes of $\sim$ form a partition of U. As there are countably many of them, and each is an open interval, we have proven that U is a union of finitely or countably many mutually disjoint open intervals.