Name: ________________________________ **Instructor:** Jeffrey Diller

Math 365: Honors Real Analysis I
Fall Semester 2003
Exam 2 (Final)
Wednesday, December 17

This Examination contains 4 problems. Counting the front cover (and blank pages), the exam consists of 8 sheets of paper.

Scores

Question	Possible	Actual
1	30	
2	60	
3	40	
4	20	
Total	100	

GOOD LUCK

1. Do each of the following (6 points each for the first three, 12 for the last).

(a) State Taylor's Theorem.

(b) State either version of the Fundamental Theorem of Calculus.

(c) Define *radius of convergence* (of a power series).

(d) Let $f : [a, b] \to$ be a function. *Carefully and in detail* state what it means for f to be Riemann integrable.

2. Four of the following nine assertions are false. Identify them and give counterexamples on the following page. Note that you do not have to justify your counterexample. (15 points for correctly identifying and giving a counterexample to each false statement.)

(a) If $\{f_n : [0,1] \to\}_{n\in}$ is a sequence of continuous functions such that $f(x) := \lim_{n\to\infty} f_n(x)$ exists for every $x \in [0,1]$, then the resulting function $f : [0,1] \to$ is continuous.

(b) If X, Y are metric spaces and $f : X \to Y$ is a function such that $f^{-1}(K)$ is closed for every closed set $K \subset Y$, then f is continuous.

(c) If $f :\to$ is differentiable at every point, then the function $f' :\to$ is continuous.

(d) If $\{f_n : [a,b] \to\}$ are continuous functions converging uniformly to a function $f : [a,b] \to$, then the function f is Riemann integrable and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f(x)\,dx.$$

(e) If $f, g :\to$ are continuous functions such that $f(x) = g(x)$ for every $x \in$, then $f \equiv g$.

(f) If $f, g :\to$ are C^∞ functions such that $f(x) = g(x)$ for every $x \in (-1,1)$, then $f(x) = g(x)$ for every $x \in$.

(g) Suppose that $\{f_n : [0,1] \to\}$ is a sequence of functions that converges pointwise but not uniformly to a function $f : [0,1] \to$. Then there exists an $\epsilon > 0$ such that for any $n \in$, we can find $x \in [0,1]$ such that

$$|f_n(x) - f(x)| \geq \epsilon.$$

(h) If $\sum_{n=0}^{\infty} c_n(x-1)^n$ is a power series that converges at the point $x = -1$, then the series also converges when $x = 2$.

(i) An increasing function $f : [0,1] \to$ is always Riemann integrable.

Counterexamples for problem 2:

3. Do two of the following three problems. If you turn in solutions to all four, I will simply grade the first three. (20 points each)

(a) Let $f :\to$ be a function and $x_0 \in$ a point such that f is continuous at x_0 and $f(x_0) > 0$. Prove that there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies that $f(x) > 0$.

(b) Use the *definition* of integrability to show that the function

$$f(x) = \begin{cases} x & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$

is Riemann integrable on the closed interval $[-1, 1]$.

(c) Prove directly from the $\epsilon - \delta$ definition (4.5 from Rudin, if that means anything) of continuity that $f(x) = 1/x$ is continuous at every point $x > 0$.

(Solution to problem 3, continued)

4. Do one of the following two problems. (20 points)

(a) State and prove the contraction mapping theorem.

(b) State and prove the mean value theorem. If your proof relies on Rolle's Theorem, then you should prove that, too.