Supplementary problems (assigned 11/24/03)

1. Solve the following initial value problems

   (a) $y' = \frac{\sin t}{y}$, $y(\pi/2) = 1$.

   (b) $(1+t^2)y' + 4ty = (1+t^2)^{-2}$, $y(1) = 0$.

2. Disillusioned about mathematics, you descend the ivory tower and set yourself the task of becoming a millionaire by age 50. The plan is simple. You will stash away money continuously from now til then at a constant rate of R\$ per year. You figure that in your new life as day trader, you can make a reliable 8% annual interest (compounded continuously, of course) on your savings. Unfortunatly, college has left you broke as well as disillusioned, so you're starting from nothing. At what rate R will you need to be saving your money?

3. Remember Newton's method? The idea is that you have an open set $U \subset$ and a C^1 function $f : U \to$. You know that $f(r) = 0$ for some point $r \in U$, and you have a decent initial guess x_0 at the location of r. Beginning with this guess, you then produce a sequence of (hopefully better) approximations of r by setting

   $$x_{n+1} = N(x_n)$$

   for every $n \in$, where $N(x) = x - f(x)/f'(x)$. Assuming that the root r of f is non-degenerate (meaning $f'(r) \neq 0$), show that there exists $\delta > 0$ such that

   - $x \in (r-\delta, r+\delta)$ implies that $N(x) \in (r-\delta, r+\delta)$;
   - the restriction of N to the interval $(x-\delta, x+\delta)$ is a contraction mapping;
   - if $x_0 \in (r-\delta, r+\delta)$, then $\lim_{n\to\infty} x_n = r$.

   The main point here is the second one. The other two follow fairly quickly once you figure this one out. To prove the second item, you'll need the Mean Value Theorem again. Also note that you'll definitely need the continuity of f' here.

4. Suppose that $f, g :^2 \to$ are continuous functions and that $f(y,t) > g(y,t)$ for all points $(y,t) \in^2$. Let $y_1, y_2 : \to$ be functions satisfying

   $$y_1' = f(y_1, t), \quad y_2' = g(y_2, t)$$

   for all $t \in$. Show that $y_1(t_0) = y_2(t_0)$ for at most one point $t_0 \in$. If such a point t_0 exists, what must be the relationship between $y_1(t)$ and $y_2(t)$ for points $t > t_0$? $t < t_0$?

5. Let $y = C(t)$ be the unique solution of the initial value problem

   $$y'' + y = 0, \qquad y(0) = 1, y'(0) = 0.$$

   It can be shown (i.e. you can assume) that C is well-defined for all $t \in$. Let $S(t) = -C'(t)$. Prove the following about C and S.

(a) $S'(t) = C(t)$ for all $t \in$.

(b) S satisfies the same differential equation as C but with initial values $S(0) = 0$, $S'(0) = 1$.

(c) $S^2(t) + C^2(t) = 1$ for all $t \in$. In particular, the ranges of S and C are contained in the interval $[-1, 1]$.

(d) C is an even function and S is odd (i.e. $C(-t) = C(t)$ and $S(-t) = -S(t)$ for all $t \in$.

(e) There exists a number $P/2 > 0$ such that $C(P/2) = 0$ but $C(t) > 0$ for all $t \in [0, P/2)$.

(f) $S(P/2) = 0$.

(g) $C(t + a) = C(t)C(a) - S(t)S(a)$ and $S(t + a) = S(t)C(a) + S(a)C(t)$ for all $t, a \in$.

(h) S and C are periodic functions with (minimal) period $2P$.

(i) The ranges of S and C are exactly equal to $[-1, 1]$.

6. Redo supplementary problem 2 from the homework assigned on 11/3/03:

As a guide, here is the solution to problem 1 from the same assignment:

Proof. Let $\epsilon > 0$ be given. Proving that the conclusion holds is equivalent to constructing a partition P of $[a, b]$ for which

$$U(P, f) - L(P, f) < \epsilon.$$

Let M be an upper bound for $|f|$ on $[a, b]$. By hypothesis we can find mutually disjoint open intervals I_j, $j = 1, \ldots, n$ covering the set S of discontinuities of f such that

$$|I_1| + \ldots + |I_n| < \epsilon/2M$$

Let us write $I_j = (a_j, b_j)$. By putting the intervals in order (and intersecting them with $[a, b]$, if necessary) we can suppose that

$$a \leq a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_n < b_n \leq b.$$

so that $Q = \{a, a_1, b_1, \ldots, a_n, b_n, b\}$ is a (not very well labeled!) partition of $[a, b]$.

For convenience, let us define $b_0 = a$, $a_{n+1} = b$. Then the condition $S \subset I_1 \cup \ldots \cup I_n$ means that f is continuous on the closed intervals $[b_j, a_{j+1}]$ for $0 \leq j \leq n$. Therefore f is integrable on each of these intervals, and we can choose a partition P_j of $[b_j, a_{j+1}]$ such that

$$U(P_j, f) - L(P_j, f) < \frac{\epsilon}{2(n+1)}.$$

Now we define our partition P to be the union of the P_j, $j = 1, \ldots, n$ (note in particular that $Q \subset P$, since every point in Q is the endpoint of one the partitions P_j). Then using the upper bound M for $|f|$ chosen above, we can estimate

$$\begin{aligned} U(P, f) &\leq U(P_0, f) + M(b_1 - a_1) + U(P_1, f) + M(b_2 - a_2) + \ldots + M(b_n - a_n) + U(P_n, f) \\ L(P, f) &\geq L(P_0, f) - M(b_1 - a_1) + L(P_1, f) - M(b_2 - a_2) + \ldots - M(b_n - a_n) + L(P_n, f). \end{aligned}$$

Therefore

$$U(P, f) - L(P, f) \leq \sum_{j=0}^{n} U(P_j, f) - L(P_j, f) + 2M \sum_{k=1}^{n} (b_k - a_k) < (n+1)\frac{\epsilon}{2n+1} + 2M\frac{\epsilon}{4M} = \epsilon.$$