Supplementary problems (assigned 11/24/03)

1. Solve the following initial value problems
   (a) $y' = \frac{\sin t}{y}$, $y(\pi/2) = 1$.
   (b) $(1+t^2)y' + 4ty = (1+t^2)^{-2}$, $y(1) = 0$.

2. Disillusioned about mathematics, you descend the ivory tower and set yourself the task of becoming a millionaire by age 50. The plan is simple. You will stash away money continuously from now til then at a constant rate of R\$ per year. You figure that in your new life as day trader, you can make a reliable 8% annual interest (compounded continuously, of course) on your savings. Unfortunatly, college has left you broke as well as disillusioned, so you're starting from nothing. At what rate R will you need to be saving your money?

3. Remember Newton's method? The idea is that you have an open set $U \subset$ and a C^1 function $f : U \to$. You know that $f(r) = 0$ for some point $r \in U$, and you have a decent initial guess x_0 at the location of r. Beginning with this guess, you then produce a sequence of (hopefully better) approximations of r by setting

   $$x_{n+1} = N(x_n)$$

   for every $n \in$, where $N(x) = x - f(x)/f'(x)$. Now assume that r is a *non-degenerate* root of f—i.e. that $f'(r) \neq 0$. Prove the following.

   - r is a fixed point of N.
   - There exists $\delta > 0$ such that $N(N_\delta(r)) \subset N_\delta(r)$
   - The (restricted) function $N : N_\delta(r) \to N_\delta(r)$ is a contraction mapping.

   (The mean value theorem will be useful in the second and third items.) What can you conclude from all this about how well Newton's method works?

4. Suppose that $f, g :^2 \to$ are continuous functions and that $f(y,t) > g(y,t)$ for all points $(y,t) \in^2$. Let $y_1, y_2 : \to$ be functions satisfying

   $$y_1' = f(y_1, t), \quad y_2' = g(y_2, t)$$

   for all $t \in$. Show that $y_1(t_0) = y_2(t_0)$ for at most one point $t_0 \in$.

5. Let $f : \to$ be a function. The *support* of f is the set

   $$K := \overline{\{x \in : f(x) \neq 0\}}$$

   Show that if f is real analytic, then K cannot be compact. Show by giving an example that K can be compact if f is merely C^∞.

6. Redo supplementary problem 2 from the homework assigned on 11/3/03:

As a guide, here is the solution to problem 1 from the same assignment:

Proof. Let $\epsilon > 0$ be given. Proving that the conclusion holds is equivalent to constructing a partition P of $[a, b]$ for which

$$U(P, f) - L(P, f) < \epsilon.$$

Let M be an upper bound for $|f|$ on $[a, b]$. By hypothesis we can find mutually disjoint open intervals I_j, $j = 1, \ldots, n$ covering the set S of discontinuities of f such that

$$|I_1| + \ldots + |I_n| < \epsilon/2M$$

Let us write $I_j = (a_j, b_j)$. By putting the intervals in order (and intersecting them with $[a, b]$, if necessary) we can suppose that

$$a \leq a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_n < b_n \leq b.$$

so that $Q = \{a, a_1, b_1, \ldots, a_n, b_n, b\}$ is a (not very well labeled!) partition of $[a, b]$.

For convenience, let us define $b_0 = a$, $a_{n+1} = b$. Then the condition $S \subset I_1 \cup \ldots \cup I_n$ means that f is continuous on the closed intervals $[b_j, a_{j+1}]$ for $0 \leq j \leq n$. Therefore f is integrable on each of these intervals, and we can choose a partition P_j of $[b_j, a_{j+1}]$ such that

$$U(P_j, f) - L(P_j, f) < \frac{\epsilon}{2(n+1)}.$$

Now we define our partition P to be the union of the P_j, $j = 1, \ldots, n$ (note in particular that $Q \subset P$, since every point in Q is the endpoint of one the partitions P_j). Then using the upper bound M for $|f|$ chosen above, we can estimate

$$\begin{aligned} U(P, f) &\leq U(P_0, f) + M(b_1 - a_1) + U(P_1, f) + M(b_2 - a_2) + \ldots + M(b_n - a_n) + U(P_n, f) \\ L(P, f) &\geq L(P_0, f) - M(b_1 - a_1) + L(P_1, f) - M(b_2 - a_2) + \ldots - M(b_n - a_n) + L(P_n, f). \end{aligned}$$

Therefore

$$U(P, f) - L(P, f) \leq \sum_{j=0}^{n} U(P_j, f) - L(P_j, f) + 2M \sum_{k=1}^{n} (b_k - a_k) < (n+1)\frac{\epsilon}{2n+1} + 2M\frac{\epsilon}{4M} = \epsilon.$$