Solutions to Homework 8

Supplementary problem 1. Let $\epsilon > 0$ be given. Proving that the conclusion holds is equivalent to constructing a partition P of $[a, b]$ for which

$$U(P, f) - L(P, f) < \epsilon.$$

Let M be an upper bound for $|f|$ on $[a, b]$. By hypothesis we can find mutually disjoint open intervals I_j, $j = 1, \dots, n$ covering the set S of discontinuities of f such that

$$|I_1| + \dots + |I_n| < \epsilon/2M$$

Let us write $I_j = (a_j, b_j)$. By putting the intervals in order (and intersecting them with $[a, b]$, if necessary) we can suppose that

$$a \le a_1 < b_1 \le a_2 < b_2 \le \dots \le a_n < b_n \le b.$$

so that $Q = \{a, a_1, b_1, \dots, a_n, b_n, b\}$ is a (not very well labeled!) partition of $[a, b]$.

For convenience, let us define $b_0 = a$, $a_{n+1} = b$. Then the condition $S \subset I_1 \cup \dots \cup I_n$ means that f is continuous on the closed intervals $[b_j, a_{j+1}]$ for $0 \le j \le n$. Therefore f is integrable on each of these intervals, and we can choose a partition P_j of $[b_j, a_{j+1}]$ such that

$$U(P_j, f) - L(P_j, f) < \frac{\epsilon}{2(n+1)}.$$

Now we define our partition P to be the union of the P_j, $j = 1, \dots, n$ (note in particular that $Q \subset P$, since every point in Q is the endpoint of one the partitions P_j). Then using the upper bound M for $|f|$ chosen above, we can estimate

$$\begin{aligned} U(P, f) &\le U(P_0, f) + M(b_1 - a_1) + U(P_1, f) + M(b_2 - a_2) + \dots + M(b_n - a_n) + U(P_n, f) \\ L(P, f) &\ge L(P_0, f) - M(b_1 - a_1) + L(P_1, f) - M(b_2 - a_2) + \dots - M(b_n - a_n) + L(P_n, f). \end{aligned}$$

Therefore

$$U(P, f) - L(P, f) \le \sum_{j=0}^{n} U(P_j, f) - L(P_j, f) + 2M \sum_{k=1}^{n} (b_k - a_k) < (n+1)\frac{\epsilon}{2n+1} + 2M\frac{\epsilon}{4M} = \epsilon.$$

Supplementary problem 2. Given $\epsilon > 0$, we again seek to construct a partition P of $[0, 1]$ satisfying

$$U(P, f) - L(P, f) < \epsilon.$$

This time, however, we let

$$S = \{x \in [0, 1] : f(x) \ge \epsilon/2\}.$$

I claim that there are only finitely many points in S. Indeed any $x \in S$ is rational, and can be written $x = p/q$ where $\gcd(p, q) = 1$ and $q \le 2/\epsilon$ (since $f(x) = 1/q$). So if $N \in$ exceeds $2/\epsilon$, then the denominator q is x can only range from 1 to N. Moreover, for fixed denominator q, the numerator of x must range between 0 and q. So all told, S contains at most $\sum_{q=1}^{N}(q+1) = (q+1)(q+2)/2$ elements.

Clearly then, we can cover S with disjoint open intervals $I_j = (a_j, b_j)$, $j = 1, \dots, n := \#S$ such that $\sum_{j=1}^n b_j - a_j < \epsilon/2$. Putting the intervals in order and intersecting them with $[0,1]$ we obtain a partition P of $[0,1]$ consisting of the points

$$a = a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n = b.$$

Since $0 \le f(x) \le 1$ for all x, we have

$$0 \le L(P,f) \le U(p,f) \le \sum_{j=1}^{n} 1(b_j - a_j) + \sum_{j=1}^{n-1} \epsilon/2(a_{j+1} - b_j).$$

The first sum is equal to the sum of the lengths of the intervals I_j and the second is no larger than $\epsilon/2$ times the length of the full interval $[0,1]$. Hence

$$U(P,f) - L(P,f) < \epsilon$$

and we conclude that f is integrable.

Supplementary problem 3.

Part a. Let $P = \{n, n+1, \dots, m, m+1\}$ be the partition of $[n, m+1]$ by integers. Then since f is decreasing, we have $a_k = \sup\{f(x) : x \in [k, k+1]\}$, and

$$\int_n^{m+1} f(x)\,dx \le U(P,f) = \sum_{k=n}^{m} a_k.$$

Similarly, using the partition $P = \{n-1, n, \dots, m\}$ of $[n-1, m]$ we obtain

$$\int_{n-1}^{m} f(x)\,dx \ge L(P,f) = \sum_{k=n}^{m} a_k.$$

Part b. Since the series consists of non-negative terms, it's enough to show that the partial sums s_n are bounded above uniformly in n. The function $f(x) = 1/x(\log x)^2$ is non-negative and decreasing, and $a_n = f(n)$ for all $n \in$, so we can use the previous item to estimate

$$s_n = a_2 + \sum_{k=3}^{n} a_k \le \int_2^n \frac{1}{x(\log x)^2}\,dx = a_2 + \frac{1}{\log 2} - \frac{1}{\log n} \le a_2 + \frac{1}{\log 2}$$

for all $n \in$. Since the right side is independent of n, the series converges.

Part c. The difference between the full series s and a given partial sum s_n can be estimated as follows.

$$s - s_n = \lim_{m\to\infty} s_m - s_n \le \lim_{n\to\infty} \int_{n-1}^{m} f(x)\,dx = \frac{1}{\log(n-1)}.$$

So for s_n to approximate s to within .01 it is sufficient that $1/\log(n-1) < .01$. That is, we need

$$n \ge 1 + e^1 00 \approx 2.68812 \times 10^{43}.$$

Assuming we had a computer capable of evaluating and adding a trillion terms a second, we'd have to wait about

$$2.68812 \times 10^{31}\text{seconds} \approx 3.11125 \times 10^{26}\text{days} \approx 8.5 \times 10^{23}\text{years}$$

to find out what the partial sum is. But really, these days, who's got a mole of years to kill anyhow?

Part d. The first part of this problem also gives us that

$$s - s_n = \lim_{m\to\infty} s_m - s_n \geq \lim_{n\to\infty} \int_n^{m+1} f(x)\,dx = \frac{1}{\log n}.$$

So to summarize, we have

$$s_n + \frac{1}{\log n} \leq s \leq s_n + \frac{1}{\log(n-1)}.$$

This means that if we choose n large enough that

$$\frac{1}{\log(n-1)} - \frac{1}{\log n} < .01$$

we can compute s_n exactly and then replace the sum of the remaining terms by $1/\log n$, knowing that this will be skewing the result by no more than .01. We could determine our n by trial and error, but with the mean value theorem we can do better. That is, the derivative of the function $1/\log x$ is $-1/x(\log x)^2$ (why is this not a surprise?), so the mean value theorem gives us a number c between $n-1$ and n such that

$$\frac{1}{\log(n-1)} - \frac{1}{\log n} = 1 \cdot \frac{1}{c(\log c)^2} \leq \frac{1}{n(\log n)^2},$$

and the latter quantity will certainly be less than .01 when $n = 100$. In fact, since $\log 25 > 2$, we can even see that $n = 25$ will suffice. A little further playing around with a calculator reveals that 15 is the largest value of n for which $1/n(\log n)^2 < .01$. So with the help of mathematica, I now estimate

$$2.114 < \frac{1}{\log 15} + \sum_{k=2}^{15} \frac{1}{k(\log k)^2} \leq \sum_{k=2}^{\infty} \frac{1}{k(\log k)^2} \leq \frac{1}{\log 14} + \sum_{k=2}^{15} \frac{1}{k(\log k)^2} < 2.124$$

Solution to #7 on Page 138. Part a. If f is already integrable on $[0,1]$, then by definition f is bounded. Let $M \in$ be an upper bound for $|f|$ on $[0,1]$. Then by Theorem 6.12 parts (c) and (d), we have for $c \in [0,1]$ that

$$\left| \int_0^1 f(x)\,dx - \int_c^1 f(x)\,dx \right| = \left| \int_0^c f(x)\,dx \right| \leq Mc.$$

So if $\epsilon > 0$ is given and $0 < c < \delta := \epsilon/M$, we have

$$\left| \int_0^1 f(x)\,dx - \int_c^1 f(x)\,dx \right| < M\delta = \epsilon.$$

Thus

$$\lim_{c\to 0} \int_c^1 f(x)\,dx = \int_0^1 f(x)\,dx.$$

In other words, the two definitions of integral for f coincide.

Part b. Consider the function $f : [0,1] \to$ defined as follows. For each $n \in^+$ and $x \in (1/(n+1), 1/n]$, we set $f(x) = n$ if n is even and $-n$ if n is odd. Then

$$\lim_{c\to 0} \int_c^1 |f(x)|\,dx = \sum_{n=1}^{\infty} n\left(\frac{1}{n} - \frac{1}{n+1}\right) = \sum_{n=1}^{\infty} \frac{1}{n+1}$$

which diverges, but

$$\lim_{c\to 0}\int_c^1 f(x)\,dx = \sum_{n=1}^{\infty}(-1)^n n\left(\frac{1}{n}-\frac{1}{n+1}\right) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}$$

which converges.