Baire's Theorem. Page 82: 22

Lemma 0.1 *Let $U, V \subset X$ be open and V be dense. Then $U \cap V$ is open and dense in U. In particular, $U \cap V \neq \emptyset$.*

Proof. Let $p \in U$ be any given point and $r > 0$ any given radius. Shrinking r if necessary, we can assume that $N_r(p) \subset U$. Since V is dense and open, we have that $p \in V'$. Therefore, there exists a point $q \in N_r(p) \cap V \subset U \cap V$. We conclude that $U \cap V$ is dense in U. $\square$

Now let $G_j, j \in \mathbf{N}$ be the open dense sets given in the problem. We will show that $\cap G_j \neq \emptyset$. By replacing X with $N_r(p)$ for some $p \in X$ and some radius $r > 0$ and replacing G_j with $G_j \cap N_r(p)$, the above lemma applied to $U = N_r(p)$ and $V = G_j$ will allow us to conclude that in fact $\cap G_j$ contains a point in $N_r(p)$. That is, $\cap G_j$ is actually dense in X.

Working inductively, we construct a sequence of radii r_n converging to zero and points p_n such that for every $n \in \mathbf{N}$,

- $\overline{N_{r_n}(p_n)} \subset G_n$;
- $N_{r_{n+1}}(p_{n+1}) \subset N_{r_n}(p_n)$

It will then follow from problem 21 that

$$\cap G_n \supset \cap \overline{N_{r_n}(p_n)} \neq \emptyset.$$

So pick p_1, r_1 so that $N_{2r_1}(p_1) \subset G_1$. Clearly, $\overline{N_{r_1}(p_1)} \subset G_1$. Given r_k, p_k satisfying the conditions above, the lemma above allows us to choose $r_{k+1} < r_k/2$ and p_{k+1} such that $N_{2r_{k+1}}(p_{k+1}) \subset N_{r_k}(p_k) \cap G_{k+1}$. Then simply by virtue of our wise choices $N_{r_{k+1}}(p_{k+1})$ satisfies both of the above conditions. The result is a decreasing sequence

$$\overline{N_{r_1}(p_1)} \supset \overline{N_{r_2}(p_2)} \supset \dots$$

of closed sets with diameters converging to zero, and the unique point in the intersection of all these sets is certainly a point in $\cap G_n$. $\square$