Solutions to Homework 10

Supplementary problem 1. Prove that $\cos(2x)$ is analytic at every point.

Solution. Let $f(x) = \cos(2x)$ and fix a point $a \in \mathbf{R}$. Let $T_n(x)$ be the nth order Taylor polynomial of f centered at a. We will show that $\lim_{n\to\infty} T_n(x) = f(x)$ for every $x \in \mathbf{R}$—i.e.

$$\lim_{n\to\infty} |f(x) - T_n(x)| = 0.$$

By Taylor's theorem, we have

$$|f(x) - T_n(x)| = \frac{|f^{n+1}(c)|}{(n+1)!}|x-a|^{n+1}$$

for some c between x and a. Moreover, the $(n+1)$st derivative of f at c is always 2^{n+1} times $\pm\cos 2c$ or $\pm\sin 2c$, so $|f^{n+1}(c)| \le 2^{n+1}$. Therefore

$$|f(x) - T_n(x)| \le \frac{(2|x-a|)^{n+1}}{(n+1)!}$$

which converges to 0 as $n \to \infty$, regardless of x and a. Hence for all $x \in \mathbf{R}$ (especially those in a neighborhood of a) $f(x)$ agrees with its Taylor series (centered at a) evaluated at x. We conclude that f is analytic at a. Since a was arbitrary, f is analytic at all points in $\mathbf{R}$. □

Supplementary problem 2 (Not entirely irrelevant sequence problem) Let $\{a_{i,j}\}_{i,j\in\mathbf{N}} \subset \mathbf{R}^+$ be a double sequence of positive numbers. Suppose that

- for each fixed $i \in \mathbf{N}$ the sequence $\{a_{i,j}\}_{j\in\mathbf{N}}$ is increasing and converges to to a number $A_i \in \mathbf{R}$; and
- $\sum_{i=0}^{\infty} A_i$ converges (call the value of the sum S).

Show that

- $\sum_{i=0}^{\infty} a_{i,j}$ converges for each fixed $j \in \mathbf{N}$ (call the value of the sum S_j); and
- $\lim_{j\to\infty} S_j = S$.

The main point here is the second item, because it involves switching two limits.

Solution. Since $|a_{i,j}| = a_{i,j} \le A_i$ and since $\sum A_i$ converges, we can apply Theorem 3.25 to conclude that $\sum_{i=0}^{\infty} a_{i,j}$ converges. So the first conclusion holds.

Now to show that $\lim_{j\to\infty} S_j = S$, let $s_n = \sum_{i=0}^{n} A_i$ and $s_{n,j} = \sum_{i=0}^{n} a_{i,j}$ be the partial sums of the series concerned. Given $\epsilon > 0$, choose $N \in \mathbf{N}$ such that that $n \ge N$ implies that

$$\sum_{i=n+1}^{\infty} A_i = |S - s_n| \le \epsilon/3.$$

It follows that

$$|S_j - s_{n,j}| = \sum_{i=n+1}^{\infty} a_{i,j} \le \sum_{i=n+1} A_i \le \epsilon/3,$$

as well.

Moreover, we can exchange a limit with a *finite* sum to obtain

$$\lim_{j\to\infty} s_{N,j} = s_N.$$

Hence, there exists $J \in \mathbf{N}$ such that $j \geq J$ implies that

$$|s_{N,j} - s_N| \leq \epsilon/3.$$

Putting these observations together gives us for $j \geq J$ that

$$|S_j - S| = |S_j - s_{N,j} + s_{N,j} - S_N + S_N - S| \leq |S_j - s_{N,j}| + |s_{N,j} - S_N| + |S_N - S| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

So $S_j \to S$, as desired. □

Solution to #2 on Page 165. By hypothesis and problem #1 (on the same page), there exists a number $M \in \mathbf{R}$ such that $|f_n(x)|, |g_n(x)| \leq M$ for all x and for all $n \in \mathbf{N}$. Hence

$$|f_n(x)g_n(x) - f(x)g(x)| \quad = \quad |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n($$

So if we are given $\epsilon > 0$, we can choose $N_1, N_2 \in \mathbf{N}$ such that $n \geq N_1$ implies that

$$|f_n(x) - f(x)| < \epsilon/2M$$

for all x, and similarly, $n \geq N_2$ implies that

$$|g_n(x) - g(x)| < \epsilon/2M$$

for all x. Let $N = \max\{N_1, N_2\}$. Then $n \geq N$ and the estimates above imply that

$$|f_n(x)g_n(x) - f(x)g(x)| < M(\epsilon/2M + \epsilon/2M) = \epsilon$$

for all x. We conclude that $\{f_n g_n\}$ converges uniformly to fg. □

Solution to #3 on page 165.

Let $f_n, g_n : \mathbf{R} \to \mathbf{R}$ be given by $f_n(x) = g_n(x) = x + 1/n$. Then both $\{f_n\}$ and $\{g_n\}$ converge uniformly to the identity function $f(x) = x$. On the other hand $f_n(x)g_n(x) = x^2 + 2x/n + 1/n^2$ converges *pointwise* to $f(x)^2$ but not uniformly. To see this note that for $x = n$, we have

$$|f_n(x)g_n(x) - f(x)^2| = |2x/n + 1/n^2| \geq 2$$

for all $n \in \mathbf{N}$. So if we take $\epsilon = 2$, there is no $N \in \mathbf{N}$ such that $n \geq N$ implies that

$$|f_n(x)g_n(x) - f(x)^2| < \epsilon$$

for *all* $x \in \mathbf{R}$.

Solution to #9 on page 166. Given $\epsilon > 0$, we must find $N \in \mathbf{N}$ such that

$$|f_n(x_n) - f(x)| < \epsilon$$

when $n \geq N$, so this is what we do: by hypothesis there exists $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies that

$$|f_n(x) - f(x)| < \epsilon/2$$

for all $x \in E$. Moreover, the limit function $f(x)$ is continuous by Theorem 7.12, so there exists $\delta > 0$ such that $|x_n - x| < \delta$ implies that

$$|f(x_n) - f(x)| < \epsilon/2$$

. Finally, since $x_n \to x$, there is $N_2 \in \mathbf{N}$ such that $n \geq N_2$ implies that $n \geq N_2$ implies that

$$|x_n - x| < \delta.$$

So if we take $N = \max\{N_1, N_2\}$, we obtain for $n \geq N$ that

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon$$

as desired. □

As for the 'converse' statement, it depends on what Rudin means by 'converse' here. I interpret it as follows: *suppose that for any $x \in E$ and any sequence $\{x_n\}$ converging to x we have that*

$$\lim_{n \to \infty} f_n(x_n) = f(x).$$

Then f_n converges uniformly to f.

This is false. As a counterexample, take for instance $f_n(x) = x/n$ and $f(x) = 0$ for all $x \in \mathbf{R}$. If $\{x_n\}$ is a convergent sequence of points with limit x, then we have

$$\lim_{n \to \infty} |f_n(x_n) - f(x)| = \lim_{n \to \infty} \left| \frac{x_n}{n} \right| \leq \lim_{n \to \infty} \frac{M}{n} = 0$$

simply because a convergent sequence of real numbers is bounded (M here is any upper bound for $\{|x_n|\}$). On the other hand taking $x = n$ for each n, we see that there is always *some* point at which $|f_n(x) - f(x)| \geq 1$. Hence f_n does not converge to f uniformly. □