Solutions to Homework 11

Problem 1. Solve the following initial value problems

1. $y' = \frac{\sin t}{y}$, $y(\pi/2) = 1$.

   Solution. Rearranging and integrating the equation gives

$$\log|y| = \int \frac{dy}{y} = \int \frac{y'}{y}\,dt = \int \sin t\,dt = -\cos t + C,$$

   where the constant C is determined by plugging in the initial condition:

$$\log|1| = -0 + C.$$

   So $C = 0$. Note that this also implies that the sign y is positive when we drop the absolute value inside the logarithm (why?). We conclude that

$$\log y(t) = -\cos t \quad \Rightarrow \quad y(t) = e^{-\cos t}.$$

2. $(1+t^2)y' + 4ty = (1+t^2)^{-2}$, $y(1) = 0$.

   Solution. First we replace the right side with zero and solve the 'homogenized' version of the equation, obtaining

$$y_h(t) = \frac{A}{(1+t^2)^2}.$$

   Then we assume that y has the form $y(t) = A(t)(1+t^2)^2$ for some unknown function $A(t)$. Plugging this back into the original differential equation leads to the following formula for A':

$$A'(t) = \frac{1}{1+t^2}.$$

   Antidifferentiating both sides yields $A(t) = \tan^{-1} t + C$. So

$$y(t) = \frac{C + \tan^{-1} t}{(1+t^2)^2}.$$

   Finally, we plug in the initial condition to compute that $C = -\pi/4$. In conclusion,

$$y(t) = \frac{-\pi/4 + \tan^{-1} t}{(1+t^2)^2}.$$

Problem 2. Disillusioned about mathematics, you descend the ivory tower and set yourself the task of becoming a millionaire by age 50. The plan is simple. You will stash away money continuously from now til then at a constant rate of R\$ per year. You figure that in your new life as day trader, you can make a reliable 8% annual interest (compounded continuously, of course) on your savings. Unfortunatly, college has left you broke as well as disillusioned, so you're starting from nothing. At what rate R will you need to be saving your money?

Solution. If $y(t)$ is the amount of money saved up (with interest) at time t, then then y satisfies the following conditions ($t = 0$ is now, $t = 30$ is age 50).

$$y(0) = 0, \quad y(30) = 10^6, \quad y' = .08y + R.$$

The differential equation is separable, and solving it gives

$$y(t) = -\frac{R}{.08} + Ae^{.08t}$$

for some constant A. Using the two additional conditions gives us

$$0 = -R/.08 + A, \quad 10^6 = -R/.08 + Ae^{2.4},$$

and these can be solved together to give $R = .08 \cdot 10^6(e^{2.4} - 1) \approx \$7981.5$ per year.

Problem 3. *This problem was mis-stated: the function f should have been C^2 instead of C^1, and the second item should have said $N(N_\delta(r)) \subset N_\delta(r)$ rather than $f(N_\delta(r)) \subset N_\delta(r)$. The upshot is that I'll let this one go without grading. The correct (I hope) statement and solution of the problem are as follows.*

Remember Newton's method? The idea is that you have a C^2 function $f : \mathbf{R} \to \mathbf{R}$. You know that $f(r) = 0$ for some point $r \in U$, and you have a decent initial guess x_0 at the location of r. Beginning with this guess, you then produce a sequence of (hopefully better) approximations of r by setting

$$x_{n+1} = N(x_n)$$

for every $n \in \mathbf{N}$, where $N(x) = x - f(x)/f'(x)$. Now assume that r is a *non-degenerate* root of f—i.e. that $f'(r) \neq 0$. Prove the following.

- r is a fixed point of N.

  Solution. $N(r) = r - f(r)/f'(r) = r - 0/f'(r) = r$. □

- There exists $\delta > 0$ such that $N(N_\delta(r)) \subset N_\delta(r)$

  Solution. Given x, the mean value theorem gives us a number c between x and r such that

  $$N(x) - N(r) = N'(c)(x - r) = \frac{f''(c)f(c)}{f(c)^2}(x - r).$$

  Moreover, since f is C^2, $f'(r) \neq 0$, and $f(r) = 0$, the function $N' = ff''/(f')^2$ is continuous in some neighborhood $N_{\delta_0}(r)$ (i.e. on any open set where $f'(r)$ does not vanish) and satisfies $N'(r) = 0$. Hence there exists $0 < \delta < \delta_0$ such that $|x - r| < \delta$ implies that

  $$|N'(x)| = |N'(x) - N'(r)| < \frac{1}{2}.$$

Putting the two displayed equations together allows us to conclude that

$$|N(x) - r| = |N(x) - N(r)| < \frac{1}{2}|x - r|$$

for all $x \in N_\delta(r)$. In particular $x \in N_\delta(r)$ implies that $N(x) \in N_{\delta/2}(r)$. That is,

$$N(N_\delta(r)) \subset N_{\delta/2}(r) \subset N_\delta(r)$$

as asserted. □

- The (restricted) function $N : N_\delta(r) \to N_\delta(r)$ is a contraction mapping.

  Solution. Let δ be as in the solution to the previous item. Then for any points $x_1, x_2 \in N_\delta(r)$, we have

  $$|N(x_1) - N(x_2)| = |N'(c)||x_1 - x_2| \le \frac{1}{2}|x_1 - x_2|,$$

  where c is between x_1 and x_2 (and therefore belongs to $N_\delta(r)$). □

(The mean value theorem will be useful in the second and third items.) What can you conclude from all this about how well Newton's method works?

Answer. r is the unique fixed point of N in the open interval $N_\delta(r)$, and if the initial guess x_0 happens to be in $N_\delta(r)$, then the sequence $x_1, x_2, \ldots$ of subsequent guesses will remain in $N_\delta(r)$ and converge to r. In short, Newton's method works if the initial guess is good enough.

Problem 4. Suppose that $f, g : \mathbf{R}^2 \to \mathbf{R}$ are continuous functions and that $f(y,t) > g(y,t)$ for all points $(y,t) \in \mathbf{R}^2$. Let $y_1, y_2 : \mathbf{R} \to \mathbf{R}$ be functions satisfying

$$y_1' = f(y_1, t), \quad y_2' = g(y_2, t)$$

for all $t \in \mathbf{R}$. Show that $y_1(t_0) = y_2(t_0)$ for at most one point $t_0 \in \mathbf{R}$.

Solution. Let $h : \mathbf{R} \to \mathbf{R}$ be the function $h(t) = y_1(t) - y_2(t)$. Then in particular, $h'(t) = f(y_1(t), t) - g(y_2(t), t)$ exists and is continuous and every point $t \in \mathbf{R}$. Also, if $t_0 \in \mathbf{R}$ is any point where $y_0 := y_1(t_0) = y_2(t_0)$, then

$$h(t_0) = 0, \quad h'(t_0) = f(y_0, t_0) - g(y_0, t_0) > 0.$$

By continuity, we see that there exists $\delta_0 > 0$ such that $h'(t) > 0$ for all $|t - t_0| < \delta_0$. That is, h is strictly increasing on $(t_0 - \delta_0, t_0 + \delta_0)$. So $h(t) > h(t_0)$ for all $t \in (t_0, t_0 + \delta_0)$ and $h(t) < h(t_0)$ for all $t \in (t_0 - \delta_0, t_0)$.

Now suppose, for the sake of obtaining a contradiction, that there is a second point at which y_1 equals y_2. We can suppose without loss of generality that this point is larger than t_0. Then it is meaningful to define

$$t_1 := \inf\{t > t_0 : h(t) = y_1(t) - y_2(t) = 0\}$$

(i.e. we're not taking the infimum of the empty set). By continuity $h(t_1) = 0$. Clearly, $t_1 \geq t_0$, so the work above implies in fact that $t_1 \geq t_0 + \delta_0 > t_0$. Repeating the arguments used on for t_0, we see that there exists $\delta_1 > 0$ such that, among other things, $h(t) < 0$ for $t \in (t_1 - \delta_1, t_1)$.

So to be perfectly specific, let us consider for example the points $s_0 = t_0 + \delta_0/2$ and $s_1 = t_1 - \delta_1/2$. Then

$$t_0 < s_0 < s_1 < t_1, \quad \text{and} \quad h(s_0) > 0 > h(s_1).$$

The intermediate value theorem therefore gives us a point $s \in (s_0, s_1) \subset (t_0, t_1)$ such that $h(s) = y_1(s) - y_2(s) = 0$. This contradicts the fact that t_1 was supposed to be the smallest root of h larger than t_0. It follows that there is *no* point other than t_0 at which $y_1 = y_2$. □

Problem 5. Let $f : \mathbf{R} \to \mathbf{R}$ be a function. The *support* of f is the set

$$K := \overline{\{x \in \mathbf{R} : f(x) \neq 0\}}$$

Show that if f is real analytic (and not the zero function), then K cannot be compact. Show by giving an example that K can be compact if f is merely C^∞.

Solution. Suppose that f is analytic and let $g(x) \equiv 0$ be the zero function, which is also analytic. If the support K of f is compact, then the set $E = \{x \in \mathbf{R} : f(x) = g(x)\}$ contains all points in the non-empty open set $\mathbf{R} - K$. This directly contradicts Theorem 8.5, which says that E has no limit points unless f and g are equal. Hence K is not compact. □

To see that C^∞ functions *can* have compact support, let $h : \mathbf{R} \to \mathbf{R}$ be the function considered in class

$$h(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ e^{-1/x} & \text{if} \quad x > 0 \end{cases}.$$

Set $f(x) = h(1-x)h(x+1)$. Then it is not hard to see that $f(x) = 0$ if $|x| \geq 1$ but $f(x) > 0$ if $|x| < 1$. So the support of f is $[-1, 1]$ which is compact.

Problem 6. Redo supplementary problem 2 from the homework assigned on 11/3/03:

Solution. Given $\epsilon > 0$, we seek to construct a partition P of $[0, 1]$ satisfying

$$U(P, f) - L(P, f) < \epsilon.$$

This time, however, we let

$$S = \{x \in [0, 1] : f(x) \geq \epsilon/2\}.$$

I claim that there are only finitely many points in S. Indeed any $x \in S$ is rational, and can be written $x = p/q$ where $\gcd(p, q) = 1$ and $q \leq 2/\epsilon$ (since $f(x) = 1/q$). So if $N \in \mathbf{N}$ exceeds $2/\epsilon$, then the denominator q is x can only range from 1 to N. Moreover, for fixed denominator q, the numerator of x must range between 0 and q. So all told, S contains at most $\sum_{q=1}^{N}(q+1) = (q+1)(q+2)/2$ elements.

Clearly then, we can cover S with disjoint open intervals $I_j = (a_j, b_j)$, $j = 1, \ldots, n := \#S$ such that $\sum_{j=1}^{n} b_j - a_j < \epsilon/2$. Putting the intervals in order and intersecting them with $[0, 1]$ we obtain a partition P of $[0, 1]$ consisting of the points

$$a = a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_n < b_n = b.$$

Since $0 \leq f(x) \leq 1$ for all x, we have

$$0 \leq L(P, f) \leq U(P, f) \leq \sum_{j=1}^{n} 1(b_j - a_j) + \sum_{j=1}^{n-1} \epsilon/2(a_{j+1} - b_j).$$

The first sum is equal to the sum of the lengths of the intervals I_j and the second is no larger than $\epsilon/2$ times the length of the full interval $[0, 1]$. Hence

$$U(P, f) - L(P, f) < \epsilon$$

and we conclude that f is integrable. □