Homework Set 4: Solutions

Supplementary Problem 1. Rudin's definition of a connected subset of a metric space is a little non-standard. Prove that his definition is equivalent to the following more standard one:

A subset E of a metric space X is connected if for every pair $U, V \subset X$ of disjoint non-empty open sets whose union contains E, we have either $E \subset U$ or $E \subset V$.

Solution. We will show that E is disconnected according to Rudin's definition if and only if E is disconnected according to the above definition.

Let us first suppose that E is disconnected according to the above definition. That is, $E \subset U \cup V$, where U and V are disjoint open sets such that $E \cap U, E \cap V \neq \emptyset$. We set

$$A \stackrel{\text{def}}{=} E \cap U, \qquad B \stackrel{\text{def}}{=} E \cap V.$$

Then it follows immediately that $A \cup B = E$ and $A, B \neq \emptyset$, To see that E is disconnected by Rudin's definition, we must show that

$$A \cap \overline{B} = \emptyset = \overline{A} \cap B.$$

Let us establish, for example, the right-hand equality (the method is exactly the same for the left-hand equality). Given $b \in B$, we have $b \in V$ by construction. Since V is open there exists $r > 0$ such that $N_r(b) \subset V$. But

$$N_r(b) \cap A \subset N_r(b) \cap U \subset V \cap U = \emptyset.$$

Therefore $b \notin \overline{A}$, and we conclude that A and B separate E—i.e. E is disconnected according to Rudin's definition.

Now we establish the reverse implication. Suppose that E is disconnected according to Rudin's definition and that A and B are sets separating E. Then for each $a \in A$, we know that $a \notin \overline{B}$, so we can choose $r = r(a) > 0$ such that $N_{r(a)}(a) \cap B = \emptyset$. We set

$$U \stackrel{\text{def}}{=} \bigcup_{a \in A} N_{r(a)/2}(a)$$

(note the factor of $1/2$). Similarly, for each $b \in B$, we have $b \notin \overline{A}$, and we can therefore choose $r(b) > 0$ so that $N_{r(b)}(b) \cap A = \emptyset$. We set

$$V \stackrel{\text{def}}{=} \bigcup_{b \in B} N_{r(b)/2}(b).$$

Then by construction, U and V are open sets whose union contains E. If we can show that they are disjoint, we will be done. So suppose that x is a point in $U \cap V$. Then there exists $a \in A$, $b \in B$ such that $x \in N_{r(a)/2}(a) \cap N_{r(b)/2}(b)$. But this means that

$$\text{dist}(a, b) \leq \text{dist}(a, x) + \text{dist}(x, b) \leq \frac{r(a)}{2} + \frac{r(b)}{2} \leq \max\{r(a), r(b)\}.$$

If, for example $r(a)$ is the larger of the two radii, then we would conclude that $b \in N_{r(a)}(a)$, contradicting the fact that $N_{r(a)}(a) \cap B = \emptyset$. Thus x does not exist, and U and V are indeed disjoint. We conclude that E is disconnected by the above definition. $\square$

Supplementary Problem 2. Let $P \subset [0,1]$ be the middle thirds Cantor set discussed in Rudin and in class.

1. Show that P is *totally disconnected*. That is, for every two points $x \neq y$ in P, there are open sets $U, V \subset \mathbf{R}$ such that $x \in U$, $y \in V$ and $P \subset U \cup V$.

   Solution: The complement of P includes every open interval of the form

   $$\left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$$

   where $k, n \in \mathbf{N}$. In particular, $\mathbf{R} - P$ includes the midpoint

   $$z_{k,n} \stackrel{\text{def}}{=} \frac{3k+1/2}{3^n}$$

   of every such interval.

   Now let $x, y \in P$ be two distinct points—say $x < y$, for example. Then choose $n \in \mathbf{N}$ so that $2 \cdot 3^{-n} < y - x$. Then if $k \in \mathbf{N}$ is the smallest integer such that $3^{-n}k > x$, we have that

   $$\left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}}\right) \subset (3^{-n}k, 3^{-n}(k+1)) \subset (x, y).$$

   In particular, $x < z_{k,n+1} < y$. Therefore the sets

   $$U \stackrel{\text{def}}{=} (-\infty, z_{k,n+1}), \qquad V \stackrel{\text{def}}{=} (z_{k,n+1}, \infty)$$

   separate P and contain x and y, respectively. □

2. Compute the sum of the lengths of the open intervals in $[0,1]-P$. Based on your computation, if we were to assign a length to the Cantor set itself, what would it have to be?

   Solution: At each stage in the creation of P we discard 2^n intervals, each of length 3^{-n-1} (starting with $n = 0$). Therefore, the total length of all intervals in the complement is

   $$\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1/3}{1-2/3} = 1.$$

   So the length of P, if it is meaningful, should be $1 - 1 = 0$.

3. Show that the sequence $\{\sin n\}_{n \in \mathbf{N}} \subset \mathbf{R}$ diverges.

   Solution: First, notice that since $\pi/2 > 1$, there is an integer n in each interval of the form $(2k\pi + \pi/4, 2k\pi + 3\pi/4)$. At that value of n, we must have $\sin n > \sqrt{2}/2$. Letting $k \to \infty$, we see that we can find arbitrarily large such n.

   Similarly, we can find arbitrarily large $n \in \mathbf{N}$ such that $n \in (2k\pi - \pi/4, 2k\pi - 3\pi/4)$ for some $k \in \mathbf{N}$, and for these values of n, we have $\sin n < -\sqrt{2}/2$.

   So if we take $\epsilon = \sqrt{2}$ and let $N \in \mathbf{N}$ be any integer, then we can find $n_1, n_2 \geq N$ such that

   $$|\sin n_1 - \sin n_2| > |\sqrt{2}/2 - (-\sqrt{2}/2)| = \sqrt{2}.$$

   This shows that $\{\sin n\}_{n \in \mathbf{N}}$ is not a Cauchy sequence and therefore does not converge. □

Page 40, # 20, solution. A connected set E need not have connected interior. For instance, take $E = \{(x, y) \in \mathbf{R}^2 : x, y \leq 0\} \cup \{(x, y) \in \mathbf{R}^2 : x, y \geq 0\}$. On the other hand, if E is connected, then $\overline{E}$ must also be connected:

Proof. We will show that $\overline{E}$ disconnected implies that E is disconnected. Indeed if $\overline{E} \subset U \cup V$, where U, V are disjoint open sets, each intersecting E, then clearly $E \subset U \cup V$ as well.

It only remains to show that $E \cap U$ and $E \cap V$ are non-empty. So choose a point $x \in \overline{E} \cap U$. Then since U is open, there exists $r > 0$ such that $N_r(x) \subset U$. But since x is either an element of E or a limit point of E, we see that $N_r(x) \cap E \neq \emptyset$. Therefore $E \cap U \neq \emptyset$ as we hoped. The proof that $E \cap V$ is non-empty is identical. □

Page 78, # 1, solution. Let $s = \lim s_n$. Then given $\epsilon > 0$, we have $N \in \mathbf{N}$ such that

$$|s_n - s| < \epsilon$$

for every $n \geq N$. But then

$$||s_n| - |s|| < |s_n - s| < \epsilon,$$

too. Therefore $\lim_{n\to\infty} |s_n| = s$. □

The converse is false, as the example $s_n = (-1)^n$ demonstrates.

Page 78, # 2, solution. First observe that

$$\begin{aligned} \lim_{n\to\infty} \sqrt{n^2 + n} - n &= \lim_{n\to\infty} (\sqrt{n^2 + n} - n) \frac{\sqrt{n^2 + n} + n}{\sqrt{n^2 + n} + n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2 + n} + n} \\ &= \lim_{n\to\infty} \frac{1}{\sqrt{1 + 1/n} + 1} = \frac{1}{1 + \lim_{n\to\infty} \sqrt{1 + 1/n}}. \end{aligned}$$

Now observe that

$$1 \leq \sqrt{1 + 1/n} \leq 1 + 1/n$$

for all $n \in \mathbf{N}$. Therefore by the squeeze theorem,

$$\lim_{n\to\infty} \sqrt{1 + 1/n} = 1.$$

We conclude that

$$\lim_{n\to\infty} \sqrt{n^2 + n} - n = \frac{1}{2}.$$