Homework Set 5: Solutions

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Note that for $n = 2m + 1$ odd, we have

$$s_{2m+1} = \frac{1}{2} + s_{2m} = \frac{1 + s_{2m-1}}{2}.$$

I.e. s_{2m+1} is the average of s_{2m-1} and 1. So if we set $a_n = s_n - 1$, we get

$$a_{2m+1} = \frac{1 + a_{2m-1} + 1}{2} - 1 = \frac{a_{2m-1}}{2}.$$

Applying this formula repeatedly gives

$$s_{2m+1} = 1 + a_{2m+1} = 1 + \frac{a_1}{2^m} = 1 - \frac{1}{2^m}$$

because $a_1 = s_1 - 1 = -1$. Turning to the terms with even indices, we compute

$$s_{2m} = \frac{s_{2m-1}}{2} = \frac{1}{2} - \frac{1}{2^m}.$$

Now let $\{s_{n_k}\}_{k \in \mathbf{N}}$ be *any* convergent subsequence with limit L. Then any further subsequence $\{s_{n_{k_\ell}}\}$ (!) must converge to L, too. So if n_k is odd for infinitely many indices $k \in \mathbf{N}$, then the work above shows that $L = 1$. And if this does not happen, then n_k must be odd for infinitely many indices, in which case our work above shows that $L = 1/2$. All told, we see that any convergent subsequence has limit equal to either 1 or 1/2. Therefore,

$$\limsup s_n = 1, \qquad \liminf s_n = 1/2.$$

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Part a: Given $\epsilon > 0$, choose $N_1 \in \mathbf{N}$ such that $|s_n - s| < \epsilon/2$ for all $n \geq N_1$. Set

$$M = \max_{n<N_1} |s_n - s|$$

and choose $N_2 \in \mathbf{N}$ so that $N_1 M < (N_2 + 1)\epsilon/2$. Then take $N = \max\{N_1, N_2\}$. If $n \geq N$, we estimate

$$\begin{aligned}
|\sigma_n - s| &= \frac{|s_0 + s_1 + \ldots s_n - (n+1)s|}{n+1} \\
&\leq \frac{|s_0 - s|}{n+1} + \ldots \frac{|s_n - s|}{n+1} \\
&< \frac{|s_0 - s|}{n+1} + \ldots \frac{|s_{N_1-1} - s|}{n+1} + \frac{(n - N_1 + 1)}{n+1}\frac{\epsilon}{2} \\
&< N_1 \frac{M}{n+1} + \frac{n - N_1 + 1}{n+1}\frac{\epsilon}{2} \\
&< \frac{N_2 + 1}{n+1}\frac{\epsilon}{2} + \frac{(n - N_1 + 1)}{n+1}\frac{\epsilon}{2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

We conclude that $\lim_{n\to\infty} \sigma_n = s$. □

Parts b and c: Let $\{a_n\}$ be some sequence of positive numbers for which $\sum a_n = m$ is finite. Then in particular, $\lim a_n = 0$. Let $s_n = k$ if $n = 2^k$ for some $k \in \mathbf{N}$, but $s_n = a_n$ otherwise. Then

$$\limsup s_n = \lim_{k\to\infty} s_{2^k} = \infty, \qquad \liminf s_n = \lim a_n = 0$$

so that $\lim_{n\to\infty} s_n$ does not exist. Moreover, given $n \in \mathbf{N}$, let K be the largest integer such that $2^K \leq n$. Then

$$\sigma_n < \frac{1}{n+1}\left(\sum_0^n a_n + \sum_{k=0}^K k\right) < \frac{1}{n+1}\left(m + \frac{K(K+1)}{2}\right) \leq \frac{m}{n+1} + \frac{K(K+1)}{2(2^K+1)}$$

which tends to zero as n (and therefore K) tends to infinity.

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Part a: First note that if $x_n > \sqrt{\alpha}$, then

$$x_{n+1} - \sqrt{\alpha} = \frac{x_n^2 - 2x_n\sqrt{\alpha} + \alpha}{2x_n} = \frac{(x_n - \sqrt{\alpha})^2}{2x_n} > 0.$$

so that $x_{n+1} > \sqrt{\alpha}$, too. Since $x_1 > \sqrt{\alpha}$, we conclude that $x_n > \sqrt{\alpha}$ for every $n \in \mathbf{N}$.

In addition

$$x_{n+1} - x_n = \frac{\sqrt{\alpha} - x_n}{2x_n} < 0,$$

so that $\{x_n\}$ is a decreasing sequence that is bounded below by $\sqrt{\alpha}$. We conclude that $L := \lim_{n\to\infty} x_n$ exists and that $L \geq \sqrt{\alpha}$.

Taking the limit of both sides in the recursion formula for x_n gives

$$L = \frac{L^2 + \sqrt{a}}{2L}$$

which, after rearranging, yields $L^2 = \alpha$. Therefore $L = \sqrt{\alpha}$. □

Part b: The formula for ϵ_{n+1} in terms of ϵ_n is just the first displayed equation in part (a). Everything else follows from $x_n > \sqrt{\alpha}$ and induction on n.

Page 82, # 20. Let $p = \lim_{i\to\infty} p_{n_i}$ be the limit of the convergent subsequence. We will show using the definition of limit that $p = \lim_{n\to\infty} p_n$. To this end, let $\epsilon > 0$ be given. By hypothesis, there exists $N_1 \in \mathbf{N}$ such that $i \geq N_1$ implies that $d(p_{n_i}, p) < \epsilon/2$. Also, since $\{p_n\}$ is Cauchy, there exists $N_2 \in \mathbf{N}$ such that $n, m \geq N_2$ implies $d(p_n, p_m) < \epsilon/2$. Now let $N = \max\{N_1, N_2\}$ and choose $i \geq N$. Then if $n \geq N$, we have

$$d(p_n, p) \leq d(p_n, p_{n_i}) + d(p_{n_i}, p) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Note that the bound on the first term in the middle comes from the fact that $n_i \geq i \geq N_2$, and the bound on the second term comes from the fact that $i \geq N_1$. □

Page 82, # 23. Let $d_n = d(p_n, q_n)$. Then we must show that the sequence $\{d_n\} \subset \mathbf{R}$ converges. Since $\mathbf{R}$ is complete, it will be enough to show this sequence is Cauchy. We prove this last fact directly from the definition of Cauchy sequence.

Let $\epsilon > 0$ be given. Then by hypothesis, there exists $N_1 \in \mathbf{N}$ such that $d(p_n, p_m) < \epsilon/2$ for all $n, m \geq N_1$. Similarly, we have $N_2 \in \mathbf{N}$ such that $d(q_n, q_m) < \epsilon/2$ for all $n, m \geq N_2$. So taking $N = \max\{N_1, N_2\}$ and arbitrary integers $n, m \geq N$, we apply the triangle inequality (four times!) to estimate

$$\begin{aligned}
|d_n - d_m| &= |d(p_n, q_n) - d(p_m, q_m)| \\
&= |d(p_n, q_n) - d(p_n, q_m) + d(p_n, q_m) - d(p_m, q_m)| \\
&\leq |d(p_n, q_n) - d(p_n, q_m)| + |d(p_n, q_m) - d(p_m, q_m)| \\
&\leq d(q_n, q_m) + d(p_n, p_m) < \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

(To see what's going on with the second last inequality, it helps to draw yourself a picture.) We conclude that $\{d_n\}$ is Cauchy, which is what we needed to prove. □