Solutions to Homework 6

Supplementary problem 1. Prove directly from Definition 4.5 that the function $f : [0,\infty) \to [0,\infty)$ given by $f(x) = \sqrt{x}$ is continuous at every point in its domain.

Solution. First consider the point 0. If $\epsilon > 0$ is given, then take $\delta = \epsilon^2$. Then $|x| < \delta$ (and $x \in [0,\infty)$) implies that

$$|\sqrt{x} - \sqrt{0}| = \sqrt{x} < \sqrt{\delta} = \epsilon.$$

So f is continuous at the point 0.

Now let $a \in (0,\infty)$ be some other point. If $\epsilon > 0$ is given, set $\delta = \sqrt{a}\epsilon$. Then for any $x \in [0,\infty)$ such that $|x - a| < \delta$, we have

$$|\sqrt{x} - \sqrt{a}| = \left|\frac{x-a}{\sqrt{x}+\sqrt{a}}\right| \le \frac{|x-a|}{\sqrt{a}} < \frac{\delta}{\sqrt{a}} = \epsilon.$$

So f is continuous at a. We conclude that f is continuous at every point in $[0,\infty)$. □

Supplementary problem 2. Prove each item in Theorem 4.4 directly from Definition 4.1—i.e. do not use the corresponding facts about limits of sequences.

Part a. Let $\epsilon > 0$ be given. By the hypotheses on f and g, we can choose $\delta_1, \delta_2 > 0$ so that for each $x \in E$,

- $0 < d(x,p) < \delta_1$ implies that $|f(x) - A| < \epsilon/2$;
- $0 < d(x,p) < \delta_2$ implies that $|g(x) - B| < \epsilon/2$.

Therefore let us set $\delta = \min\{\delta_1, \delta_2\}$. If $x \in E$ satisfies $0 < d(x,p) < \delta$, then the triangle inequality gives

$$|(f+g)(x) - (A+B)| \le |f(x) - A| + |g(x) - B| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence $\lim_{x\to p}(f+g)(x) = A + B$.

Part b. Let $\epsilon > 0$ be given. By hypothesis on f and g, we can choose $\delta_1, \delta_2 > 0$ so that for each $x \in E$,

- $0 < d(x,p) < \delta_1$ implies that $|f(x) - A| < \epsilon/4|B|$;
- $0 < d(x,p) < \delta_2$ implies that $|g(x) - B| < \min\{\epsilon/2|A|, |B|\}$ (in particular, $|g(x)| < 2|B|$).

Therefore let us set $\delta = \min\{\delta_1, \delta_2\}$. If $x \in E$ satisfies $0 < d(x,p) < \delta$, then

$$\begin{aligned} |(fg)(x) - AB| &\le |f(x)g(x) - Ag(x)| + |Ag(x) - AB| < 2|B||f(x) - A| + |A||g(x) - B| \\ &< 2|B|\frac{\epsilon}{4|B|} + |A|\frac{\epsilon}{2|A|} = \epsilon. \end{aligned}$$

Hence $\lim_{x\to p}(fg)(x) = AB$.

Part c. Let $\epsilon > 0$ be given. By hypothesis on f and g, we can choose $\delta_1, \delta_2 > 0$ so that for each $x \in E$,

- $0 < d(x,p) < \delta_1$ implies that $|f(x) - A| < |B|\epsilon/4$;
- $0 < d(x,p) < \delta_2$ implies that $|g(x) - B| < \min\{\epsilon, |B|^2\epsilon/|A|\}$ (in particular, $|g(x)| > |B|/2$).

Therefore let us set $\delta = \min\{\delta_1, \delta_2\}$. If $x \in E$ satisfies $0 < d(x,p) < \delta$, then

$$\begin{aligned} |(f/g)(x) - A/B| &= \left|\frac{f(x)B - Ag(x)}{g(x)B}\right| \leq \frac{2}{|B|^2}|f(x)B - Ag(x)| \\ &\leq \frac{2}{|B|^2}(|B||f(x) - A| + |A||g(x) - B|) < \frac{2}{|B|^2}\left(|B|\frac{|B|\epsilon}{4} + |A|\frac{|B|^2\epsilon}{4|A|}\right) = \epsilon. \end{aligned}$$

Hence $\lim_{x\to p}(f/g)(x) = A/B$. □

Solution to #2 on Page 98. By Theorem 3.2d in Rudin, we can choose a sequence $\{x_n\} \subset E$ converging to x (if $x \in E$, then we can take $x_n = x$ for all $n \in \mathbf{N}$). Then by continuity of f and Theorem 4.2, we see that

$$f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n).$$

Since $f(x_n) \in f(E)$ by definition, we conclude that $f(x) \in \overline{E}$. And since $x \in \overline{E}$ was arbitrary, $f(\overline{E}) \subset \overline{f(E)}$.

To see that equality need not hold, consider $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = e^x$. Then $f(\overline{R}) = f(\mathbf{R}) = (0,\infty) \neq \overline{(0,\infty)} = [0,\infty)$. □

Solution to #6 on Page 98. Let $E \subset \mathbf{R}$ be compact, $f : E \to \mathbf{R}$ be a function on E, and $G = \{(x, f(x)) \in \mathbf{R}^2 : x \in E\}$ be the graph of f.

Suppose first that f is continuous. Then by Theorem 4.10a, so is the function $F : E \to \mathbf{R}^2$ given by $F(x) = (x, f(x))$. As E is compact, we conclude that $F(E) = G$ is compact, as well.

Now suppose instead that G is compact. Again by Theorem 4.10a, it is enough to show that the map F defined in the previous paragraph is continuous. To this end, let $K \subset \mathbf{R}^2$ be closed, and consider $F^{-1}(K) = F^{-1}(K \cap G) = \pi(K \cap G)$ where $\pi : \mathbf{R}^2 \to \mathbf{R}$ is the continuous map $(x,y) \to x$. Now $K \cap G$ is a closed subset of a compact set and therefore compact. Hence $\pi(K \cap G)$ is compact and therefore closed. That is, the preimage of a closed set under F is closed, and F is therefore continuous. We conclude that f is continuous. □

Solution to #10 on Page 99. Suppose by way of contradiction that f is not uniformly continuous on X. That is, there exists $\epsilon > 0$ such that for *any* $\delta > 0$, we can find points $p, q \in X$ such that $d(p,q) < \delta$ while $d(f(p), f(q)) \geq \epsilon$. Let us fix this ϵ, and choose points $p = p_n$, $q = q_n$ as above for each δ of the form $1/n$, $n \in \mathbf{N}$. In other words, for each natural number n, we have

$$0 < d(p_n, q_n) < 1/n, \qquad d(f(p_n), f(q_n)) \geq \epsilon.$$

In particular $\lim_{n\to\infty} d(p_n, q_n) = 0$.

Now since X is compact, we can apply Theorem 3.6a from Rudin to obtain a subsequence $\{p_{n_k}\} \subset \{p_n\}$ that converges to some point $p \in X$. On the other hand $\lim_{k\to\infty} d(p_{n_k}, q_{n_k}) \to 0$, so it follows that $\lim q_{n_k} = p$, too. Therefore by continuity

$$\lim_{k\to\infty} f(p_{n_k}) = f(p) = \lim_{k\to\infty} f(q_{n_k}).$$

In particular,

$$\lim_{k \to \infty} d(f(p_{n_k}), f(q_{n_k})) = 0$$

This contradicts the fact that $d(f(p_{n_k}), f(q_{n_k})) \geq \epsilon$ for all $k \in \mathbf{N}$. Therefore, it must be the case that we started on the wrong hypothetical foot and that f must be uniformly continuous on X after all. □

Solution to #14 on Page 100. Let $I = [a, b]$. If $f(a) = a$ or $f(b) = b$, we are already done, so we can assume (because $f(I) \subset I$) that $f(a) > a$ and $f(b) < b$. Consider the function $g(x) = f(x) - x$, which is also continuous. Then by our assumptions, $g(a) > 0$ and $g(b) < 0$. So by the intermediate value theorem, there exists $x \in (a, b)$ such that $g(x) = 0$—i.e. $f(x) = x$. □