Homework Set 7: Solutions

Page 100, # 17, solution. Let x be a point at which f has a simple discontinuity. Then exactly one of the following is true.

1. $f(x+) < f(x-)$;
2. $f(x+) > f(x-)$;
3. $f(x+) = f(x-) < f(x)$;
4. $f(x+) = f(x-) > f(x)$.

It will be enough to show that each of these things, taken separately, occurs at no more than countably many values of x.

So let us first consider those x as in case (1). By the density property we can choose $p \in \mathbf{Q}$ such that $f(x+) < p < f(x-)$. Then we set

$$\epsilon = \min\{f(x-) - p, p - f(x+)\}$$

By definition of the left and right hand limits $f(x-)$ and $f(x+)$, there exists $\delta > 0$ so that

- $x - \delta < t < x$ implies $|f(x) - f(x-)| < \epsilon$ (so, in particular, $f(x) < f(x-) + \epsilon \leq p$);
- $x < t < x + \delta$ implies $|f(x) - f(x+)| < \epsilon$ (so, in particular, $f(x) > f(x+) - \epsilon \geq p$).

Invoking the density property again, we can choose $q, r \in \mathbf{Q}$ such that

$$x - \delta < q < x < r < x + \delta,$$

thereby associating to x a triple $(p, q, r) \in \mathbf{Q}^3$ such that

- $q < x < r$,
- $f(t) < p$ for all $t \in (q, x)$, and
- $f(t) > p$ for all $t \in (x, r)$.

I claim now that if x' is another point at which f has a simple discontinuity of type (1), then (at least one member of) the triple (p', q', r') associated to x' differs from the triple (p, q, r) associated to x. To see this suppose that $x' < x$ (the case $x < x'$ is identical) but $(p', q', r') = (p, q, r)$. Then $q < x' < x < r$, so if t is any point between x' and x, we have $f(t) > p$ because $x' < t < r$ but $f(t) < p$ because $q < t < x$. Since this is impossible, it must be the case that $(p', q', r') \neq (p, q, r)$.

In summary we have defined an injective function from the set of points where f has a simple discontinuity of type (1) into $\mathbf{Q}^3$. Since the latter set is countable, we conclude that f has no more than countably many simple discontinuities of type (1).

The case of simple discontinuities of type (2) is handled in a completely analogous fashion. Dealing with simple discontinuities of type (3) differs only slightly in that we choose $(p, q, r) \in \mathbf{Q}^3$ such that $f(t) < p$ for all $t \in (q, x) \cup (x, r)$. Finally simple discontinuities of type (4) are dealt with in the same way as those of type (3). □

Page 101, # 23, solution. Beginning with a convex function $f : (a, b) \to \mathbf{R}$, let us first prove that $g \circ f$ is also convex whenever $g : (c, d) \to \mathbf{R}$ is a convex increasing function whose domain includes the range of f. If $x < y$ and $\lambda \in (0, 1)$, then by definition of continuity, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

Thus since g is increasing, we obtain

$$g \circ f(\lambda x + (1-\lambda)y) \leq g(\lambda f(x) + (1-\lambda)f(y)).$$

Finally, applying the fact that g is convex to the right side of this inequality gives

$$g \circ f(\lambda x + (1-\lambda)y) \leq \lambda g(f(x)) + (1-\lambda)g(f(y)).$$

We conclude that $g \circ f$ is a convex function. Ta da!

Now we establish the 'slope inequality' given in the problem for f, because it will be useful in proving that f is continuous. That is, if $s < t < u$ are numbers in (a, b), we will prove that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}$$

Since t is between s and u, there exists $\lambda \in (0, 1)$ such that

$$t = \lambda s + (1-\lambda)t,$$

which implies that

$$t - s = (1-\lambda)(u - s)$$

Moreover, by definition of convexity, we have

$$f(t) \leq \lambda f(s) + (1-\lambda)f(u),$$

which, after subtracting $f(s)$ from both sides becomes

$$f(t) - f(s) \leq (1-\lambda)(f(u) - f(s)) \leq (t - s)\frac{f(u) - f(s)}{u - s}.$$

Dividing through by $t - s$ then gives the first inequality above. The proof of the second inequality is similar. Cha-ching!

Now let $x \in (a, b)$ be any given point. We will show that f is continuous at x. Note that by the inequalities we just proved, the function

$$m(t) := \frac{f(t) - f(x)}{t - x}$$

is increasing in $t \neq x$ (verifying that $m(t_1) \leq m(t_2)$ for $t_1 < t_2$ requires applying our inequality for $s < t < u$ to each of the three cases $x < t_1 < t_2$, $t_1 < x < t_2$, and $x < t_1 < t_2$).

So fix numbers $A < x < B$ in (a, b). Then $m(A) \leq m(t) \leq m(B)$ for all $t \in (A, B)$ not equal to x. In particular, taking $C = \max\{|m(A)|, |m(B)|\}$, we see that $|m(t)| \leq C$ for all $t \in (A, B)$. That is,

$$|f(t) - f(x)| \leq C|t - x|.$$

Now if $\epsilon > 0$ is given, we take $\delta = \min\{\epsilon/C, x - A, B - x\}$. Then $0 < |t - x| < \delta$ implies that $A < t < B$, and therefore

$$|f(t) - f(x)| \leq C|t - x| < C\delta \leq \epsilon.$$

This shows that $\lim_{t \to x} f(t) = f(x)$—i.e. f is continuous at x. As x is arbitrary f is continuous on (a, b). □

Page 114, # 6, solution. Since f is differentiable, so is g. We will show that $g'(x) \geq 0$ for every $x > 0$. Then if $0 \leq x_1 < x_2$, the mean value theorem gives us a number $c \in (x_1, x_2)$ such that

$$g(x_2) - g(x_1) = g'(c)(x_2 - x_1) \geq 0,$$

so that g is increasing, as desired.

Now

$$g'(x) = \frac{xf'(x) - f(x)}{x^2}.$$

Moreover, another application of the mean value theorem gives us $c \in (0, x)$ such that

$$\frac{f(x)}{x} = \frac{f(x) - f(0)}{x - 0} = f'(c).$$

But f' is an increasing function, so

$$\frac{f(x)}{x} \leq f'(x).$$

Rearranging this and using the fact that $x > 0$ gives

$$\frac{xf'(x) - f(x)}{x^2} \geq 0.$$

We conclude that $g'(x) \geq 0$, and therefore g is increasing. □

Page 114, # 9, solution. Yes, it does. Let $\{x_n\} \subset \mathbf{R} - \{0\}$ be any sequence of points converging to 0. The mean value theorem gives us a second sequence $\{c_n\}$ such that for every $n \in \mathbf{N}$, c_n is between x_n and 0 (so by the Squeeze Theorem $c_n \to 0$) and

$$\frac{f(x_n) - f(0)}{x_n - 0} = f'(c_n).$$

It follows that

$$\lim_{n\to\infty} \frac{f(x_n) - f(0)}{x_n - 0} = \lim_{n\to\infty} f'(c_n) = 3.$$

Since the sequence $\{x_n\}$ was arbitrary, we conclude from Theorem 4.2 that

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = 3.$$

□

Page 115, # 13abcd. Solution In all cases, it is only necessary to verify statements at $x = 0$ (or in part (c), in a neighborhood of $x = 0$).

(a) If $a = 0$, then we have already observed in class that $\lim_{x\to 0} f(x)$ does not exist, so f cannot be continuous at $x = 0$. if $a < 0$, then f is not even bounded near 0, so f is not continuous at $x = 0$. If $a > 0$, on the other hand, then $|f(x)| < |x|^a \to 0$ as $x \to 0$. So the squeeze theorem implies that $\lim_{x\to 0} f(x) = 0 = f(0)$, and f is continuous at 0.

(b) By definition of the derivative, we have

$$f'(0) = \lim_{x\to 0} x^{a-1} \sin(x^{-c})$$

which, as we observed in part (a), exists if and only if $a - 1 > 0$.

(c) Note that by part (b) we must have $a > 1$. For $x \neq 0$, we have

$$f'(x) = ax^{a-1}\sin(x^{-c}) + cx^{a-c-1}\cos(x^{-c}).$$

The first term is bounded since $a > 1$, and the second is bounded if and only if $a \geq c+1$.

(d) This follows immediately from parts (a) (which works for the cosine as well as the sine function) and (c).

Page 115, # 14. Solution Let $f : (a,b) \to \mathbf{R}$ be a differentiable a function. Supposing first that f' is monotonically increasing, we will show that f is convex. Given numbers $x < y$ in (a,b) and $\lambda \in (0,1)$, let

$$z = \lambda x + (1-\lambda)y.$$

Then the mean value theorem gives us $c_1 \in (x,z)$, $c_2 \in (z,y)$ such that

$$f(z) - f(x) = f'(c_1)(z-x), \qquad f(y) - f(z) = f'(c_2)(y-z).$$

Rewriting z in terms of x, y, and λ on the right sides of both equations, and using the fact that $f'(c_2) \geq f'(c_1)$ gives

$$\lambda(f(z) - f(x)) \leq (1-\lambda)(f(y) - f(z)),$$

which, upon solving for $f(z)$, yields

$$f(z) \leq \lambda f(x) + (1-\lambda)f(y).$$

This shows that f is convex.

Now let us begin again supposing that f is convex, and trying to show that f' is monotonically increasing. That is, if $x < y$ are two points in (a,b), we seek to prove that $f'(x) \leq f'(y)$. To do this let t be any point strictly between x and y. Then our 'slope inequality' from page 101/# 23 tells us that

$$\frac{f(t) - f(x)}{t - x} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(y) - f(t)}{y - t}.$$

Letting $t \to x$ in the left and middle expressions gives

$$f'(x) \leq \frac{f(y) - f(x)}{y - x}.$$

Letting $t \to y$ in the middle and right expressions gives

$$\frac{f(y) - f(x)}{y - x} \leq f'(y).$$

Combining the two inequalities gives $f'(x) \leq f'(y)$, as desired.

Finally, if f'' exists on (a,b), we note that f' is increasing if and only if $f''(x) \geq 0$ at every x. So by our work above, f is convex if and only if f'' is non-negative on (a,b). □

Page 116,# 19. Solution: For parts (a) and (b) it is useful to note that

$$\frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = \frac{f(\beta_n) - f(0)}{\beta_n - 0}\frac{\beta_n}{\beta_n - \alpha_n} - \frac{f(\alpha_n) - f(0)}{\alpha_n - 0}\frac{\alpha_n}{\beta_n - \alpha_n}$$

Therefore, if we compare with the derivative of f at zero, we can use the fact that

$$f'(0) = f'(0)\frac{\beta_n}{\beta_n - \alpha_n} - f'(0)\frac{\alpha_n}{\beta_n - \alpha_n}$$

together with the triangle inequality to obtain

$$\left|\frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} - f'(0)\right| \leq \left|\frac{f(\beta_n) - f(0)}{\beta_n - 0} - f'(0)\right|\left|\frac{\beta_n}{\beta_n - \alpha_n}\right| + \left|\frac{f(\alpha_n) - f(0)}{\alpha_n - 0} - f'(0)\right|\left|\frac{\alpha_n}{\beta_n - \alpha_n}\right|$$

Therefore if both $\{\beta_n/(\beta_n - \alpha_n)\}$ and $\{\alpha_n/(\beta_n - \alpha_n)\}$ are bounded, we conclude that

$$\lim_{n\to\infty}\left|\frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} - f'(0)\right| = 0.$$

We now deal with parts (a) and (b) in light of this discussion.

(a) If $\alpha_n < 0 < \beta_n$, then $\beta_n - \alpha_n$ is larger than both $|\beta_n|$ and $|\alpha_n|$. Hence,

$$\left|\frac{\beta_n}{\beta_n - \alpha_n}\right|, \left|\frac{\alpha_n}{\beta_n - \alpha_n}\right| < 1$$

for every $n \in \mathbf{N}$. It follows immediately, then, from the discussion above that

$$\lim_{n\to\infty}\frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = f'(0).$$

□

(b) By assumption, there exists $C \in \mathbf{R}$ such that $|\beta_n/(\beta_n - \alpha_n)| \leq C$ for all $n \in \mathbf{N}$. Thus

$$\left|\frac{\alpha_n}{\beta_n - \alpha_n}\right| \leq \left|\frac{\alpha_n - \beta_n}{\beta_n - \alpha_n}\right| + \left|\frac{\beta_n}{\beta_n - \alpha_n}\right| \leq 1 + C$$

for every $n \in \mathbf{N}$. It therefore follows again from the discussion preceding part (a) that

$$\lim_{n\to\infty}\frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = f'(0).$$

□

(b) By the mean value theorem, we have for every $n \in \mathbf{N}$ a number $c_n \in (a_n, b_n)$ such that

$$f'(c_n) = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}$$

Since $a_n, b_n \to 0$, it follows from the Squeeze Theorem that $c_n \to 0$. And since we are assuming now that f' is continuous at 0, we have

$$\lim_{n\to\infty}\frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = \lim_{n\to\infty} f'(c_n) = f'(\lim_{n\to\infty} c_n) = f'(0),$$

as advertised. □

Page 117,# 22abc. Solution:

(a) Suppose by way of contradiction that $x < y$ are distinct fixed points of f. Then the mean value theorem gives us $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x} = \frac{y - x}{y - x} = 1$$

contrary to our assumption that f' is never equal to 1. Therefore, f has at most one fixed point. $\square$

(b) Setting $f(t) = t$ for this particular function f gives us that

$$e^t = -1,$$

which is impossible. Therefore, f has no fixed points. On the other hand

$$f'(t) = 1 - \frac{e^t}{(1 + e^t)^2}.$$

Moreover, since $e^t > 0$ for every $t \in \mathbf{R}$,

$$0 < \frac{e^t}{(1 + e^t)^2} < \frac{e^t + 1}{(1 + e^t)^2} = \frac{1}{1 + e^t} < 1.$$

That is, $1 > f'(t) > 0$ for all $t \in \mathbf{R}$. $\square$

(c) Let $x_1 \in \mathbf{R}$ be any point and $\{x_n\}$ be the sequence determined by setting $x_{n+1} = f(x_n)$ for every $n \in \mathbf{N}$. Then by the mean value theorem

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| = |f'(c)||x_n - x_{n-1}|$$

for some c between x_n and x_{n-1}. In particular,

$$|x_{n+1} - x_n| \leq A|x_n - x_{n-1}|,$$

where A is the constant given in the problem. Applying this inequality inductively gives

$$|x_{n+1} - x_n| \leq A^{n-1}|x_2 - x_1|.$$

We will use this to show that $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$ be given and choose $N \in \mathbf{N}$ large enough that $\frac{A^N}{1-A} < \epsilon/C$, where $C := |x_2 - x_1|$. Then if $m > n \geq N$ we have

$$|x_m - x_n| = \left| \sum_{j=n}^{m-1} (x_{j+1} - x_j) \right| \leq \sum_{j=n}^{m-1} |x_{j+1} - x_j| \leq C \sum_{j=n}^{m-1} A^j \leq C \sum_{j=n}^{\infty} A^j = C \frac{A^n}{1 - A} < \epsilon.$$

This proves that $\{x_n\}$ is a Cauchy sequence. Since $\mathbf{R}$ is complete, we have $x \in \mathbf{R}$ such that $\lim x_n = x$. Since f is continuous, we also have

$$f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x.$$

So x is a fixed point for f. By part (a) there can be no other fixed point for f, so x does not depend at all on our initial choice of x_1 $\square$

Alternative Proof (taken shamelessly from Jenista's hwk): Note that by hypothesis, we have

$$|f(x) - f(0)| \leq A|x - 0| = A|x|$$

for every $x \in \mathbf{R}$. So suppose, for instance that $f(0) > 0$ (the case $f(0) < 0$ is similar, and if $f(0) = 0$, then 0 itself is the fixed point). Then the above inequality gives us for $x > 0$ that

$$f(x) - f(0) \leq Ax$$

(i.e. the graph of f stays below the line $y = f(0) + Ax$ which has slop less than 1). Therefore,

$$f(x) - x \leq f(0) + (A - 1)x$$

for $x > 0$. In particular, since $A - 1 < 0$, we have $f(b) - b < 0$ for b large. But $f(x) - x$ is continous and $f(0) - 0 > 0$. So by the intermediate value theorem, there exists $a \in (0, b)$ such that $f(a) - a = 0$. We conclude that there exists a fixed point $x = a$ (which is unique by part (a)).

Now if $x = x_1$ is some other point and $x_{n+1} = f(x_n)$, we have

$$|x_{n+1} - a| = |f(x_n) - f(a)| = |f'(c)||x_n - a|$$

for some c between x_n and a by the Mean Value Theorem. But this means that

$$|x_{n+1} - a| \leq A|x_n - a| \leq \ldots \leq A^n|x_1 - a|$$

for all $n \in \mathbf{N}$. Since $\lim_{n\to\infty} A^n = 0$, we conclude that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = a.$$