Solutions to Homework 9

Supplementary problem 1. (Leftover differentiation problem) Suppose that $f : (a, b) \to \mathbf{R}$ is a convex function that is differentiable at every $x \in (a, b)$. Show that f' is continuous. (Hint: you can, of course, use the results of previous homework problems about convexity; moreover, there is a theorem in the book that makes this problem much easier—for once, it is not the mean value theorem or the chain rule.)

Solution. Since f is convex, $f' : (a, b) \to \mathbf{R}$ is an increasing function. Suppose for the sake of obtaining a contradiction that f' fails to be continuous at $x \in (a, b)$. Then by Theorem 4.29, the left and right hand limits of f' exist at x, and

$$f'(x-) = \sup_{y<x} f(y) \leq f(x) \leq \inf_{z>x} f'(z) = f'(x+).$$

As we are assuming that f' is discontinuous at x, one of the inequalities in this display must be strict—without loss of generality, let us suppose that $f'(x-) < f'(x)$.

Therefore, for any $y \in (a, x)$ and any $t \in (f'(x-), f'(x))$ we have

$$f'(y) < t < f'(x).$$

But f is differentiable at every point in (a, b), so by Theorem 5.12 there exists $s \in (y, x)$ such that $f'(s) = a$. But $s < x$ also implies that $f'(s) < f'(x-)$ so that $f'(s) < a$, too—a contradiction. We conclude that f' is continuous at x after all. □

Supplementary problem 2 The function $1/t$ is continuous on $(0, \infty)$. Therefore the function

$$f(x) = \int_1^x \frac{dt}{t}.$$

is well-defined for all $x \in (0, \infty)$. Prove each of the following about f.

- f is differentiable at every point and strictly increasing.

  Proof. By the fundamental theorem of calculus (6.20), $f'(x) = 1/x$ for every $x \in (0, \infty)$, so f is differentiable at every point. Moreover, for every $0 < x_1 < x_2$, the mean value theorem gives us $c \in (x_1, x_2)$ such that

  $$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = \frac{x_2 - x_1}{c} > 0.$$

  So f is a strictly increasing function. □

- $f(xy) = f(x) + f(y)$ for every $x, y \in (0, \infty)$.

  Proof. Fix any $y \in (0, \infty)$ and set $g(x) = f(xy) - (f(x) + f(y))$. Then by the first part of this problem and the chain rule, we have

  $$g'(x) = yf'(xy) - f'(x) = 1/x - 1/x = 0.$$

for every $x \in (0, \infty)$. Moreover, $g(1) = f(y) - f(1) - f(y) = 0$ since $\int_1^1 dt/t = 0$.

So if $x \in (0, \infty)$, the mean value theorem gives us c between 1 and x such that

$$g(x) = g(x) - g(1) = g'(c)(x - 1) = 0.$$

That is, $f(xy) - (f(x) + f(y)) = 0$ for all $x, y \in (0, \infty)$. □

- $f(x^t) = tf(x)$ for all $t \in \mathbf{R}, x \in (0, \infty)$. Remember the problem from the first chapter in which x^t was defined for any real t—the idea was to do it first for $t \in \mathbf{Z}$, then for $t \in \mathbf{Q}$, and then, using supremums, for $t \in \mathbf{R}$.

Proof. For $t = k \in \mathbf{N}$, we have

$$f(x^k) = f(x \cdot x \cdot \ldots x) = f(x) + f(x) + \cdots + f(x) = kf(x),$$

by repeated application of the second part of this problem. Now suppose that $t = 1/k$ for some non-zero $k \in \mathbf{N}$. Then $(x^t)^k = x$, so by the previous display $f(x) = kf(x^t)$. In other words

$$f(x^t) = \frac{1}{k} f(x) = tf(x)$$

once again. Now if $t = p/q$ is an arbitrary rational number, we have

$$f(x^t) = f(x^(p/q)) = pf(x^{1/q}) = p/qf(x) = tf(x),$$

yet again. Now if $t \in (0, \infty)$ is irrational, we have (from page 22: 6c) by definition that

$$\begin{aligned} f(x^t) &= f(\sup\{x^s : s \in \mathbf{Q}, s < t\}) = \sup\{f(x^s) : s \in \mathbf{Q}, s < t\} \\ &= \sup\{sf(x) : s \in \mathbf{Q}, s < t\} = tf(x). \end{aligned}$$

Note that we are allowed to move the supremum past f because f is continuous (so $f(x^t) = f(x^t+)$) and increasing (so $f(x^t+) = \sup_{s<t} f(x^s)$).

Finally, we consider negative values of t. By the second part of this problem we have

$$0 = f(1) = f(x \cdot x^{-1}) = f(x) + f(x^{-1})$$

for any $x \in (0, \infty)$—i.e. the statement is true for $t = -1$. So for arbitrary $t < 0$, we have

$$f(x^t) = -f(x^{-t}) = -(-t)f(x) = tf(x)$$

since $-t > 0$. This concludes the proof. □

- $f(0, \infty) = \mathbf{R}$. In particular, there is a unique number $d \in (1, \infty)$ such that $f(d) = 1$.

Proof. First note that $f(2) = \int_1^2 dt/t > 0$ since $1/t > 0$ for all $t \in [1, 2]$. Therefore, if $y \in \mathbf{R}$ is given, we have $f(2^k) = kf(2) > y > -kf(2) = f(2^{-k})$ for $k \in \mathbf{N}$ large enough. But f is continuous (because f is differentiable), so the intermediate value theorem gives us a point $x \in (2^{-k}, 2^k)$ such that $f(x) = y$. That is, y belongs to the range of f. As y was arbitrary, the range of f is all of $\mathbf{R}$.

In particular, we have $f(d) = 1$ for some $d > 1$. And d is unique because f is strictly increasing: $f(d') > 1$ for all $d' > d$ and $f(d') < 1$ for all $d' < d$.

- f is an invertible function and that $f^{-1}(y) = d^y$ for all $y \in \mathbf{R}$.

  Proof. Since f is strictly increasing, f is injective. Together with the previous part of this problem, this tells us that $f : (0, \infty) \to \mathbf{R}$ is a bijection and therefore invertible. Let $g : \mathbf{R} \to (0, \infty)$ be the inverse function. Then

  $$f(g(t)) = t = t \cdot 1 = tf(d) = f(d^t)$$

  for all $t \in \mathbf{R}$. And since f is injective, this implies that

  $$g(t) = d^t$$

  for all $t \in \mathbf{R}$. □

Solution to #10abc on Page 138.

Part a. If $v = 0$, the inequality is trivial, so fix $v > 0$. Consider the function $h : [0, \infty) \to \mathbf{R}$ given by

$$h(t) = \frac{t^p}{p} - tv + \frac{v^q}{q}.$$

We will be done if we can show that h is non-negative. So suppose $h(t) < 0$ for some $t \in [0, \infty)$. Since $h(0) = v^q/q \geq 0$ and $\lim_{t\to\infty} h(t) = \infty$, this means that there exists $x \in \mathbf{R}$ such that $h(s) < 0$ and $h(s) \leq h(t)$ for all $t \in [0, \infty)$. In particular

$$0 = h'(s) = s^{p-1} - v,$$

so $s = v^{1/(p-1)} = v^{q/p}$ (since $1/p + 1/q = 1$). Plugging this value of s back into h gives

$$h(s) = \frac{v^q}{p} - v^{q/p+1} + \frac{v^q}{q} = v^q - v^q = 0.$$

So in fact the minimum value of h is no less than 0, and it follows that $h(t) \geq 0$ for all $t \in [0, \infty)$. That is,

$$\frac{t^p}{p} + \frac{v^q}{q} \geq tv$$

for all $t, v \in [0, \infty)$.

Finally, note that the above work shows that $h(t)$ is minimal and equal to zero if and only if $t = v^{q/p}$—i.e. if and only if $t^p = v^q$. □

Part b. For every $x \in [a, b]$ we have

$$\frac{f(x)^p}{p} + \frac{g(x)^q}{q} \geq f(x)g(x).$$

by part (a). Theorem 6.12b therefore implies that

$$\int_a^b f(x)g(x)\,dx \leq \int_a^b \frac{f(x)^p}{p}\,dx + \int_a^b \frac{g(x)^q}{q}\,dx = \frac{1}{p} + \frac{1}{q} = 1.$$

Part c. Let I_1 and I_2 denote the integrals of $|f|^p$ and $|g|^q$, respectively, on $[a, b]$. Then

$$\int_a^b \left(\frac{|f|}{I_1^{1/p}}\right)^p dx, \int_a^b \left(\frac{|g|}{I_1^{1/q}}\right)^q dx = 1.$$

So we can apply part (b):

$$\int_a^b \frac{|f(x)|}{I_1^{1/p}} \frac{|g(x)|}{I_2^{1/q}} dx \leq 1,$$

which rearranges to give

$$\left|\int_a^b f(x)g(x)\, dx\right| \leq \int_a^b |f(x)||g(x)|\, dx \leq I_1^{1/p} I_2^{1/q}.$$

□

Solution to #1 on page 165.

Let $\{f_n : X \to \mathbf{R}\}_{n\in\mathbf{N}}$ be a uniformly bounded sequence of functions from a metric space X into $\mathbf{R}$. Then for each $n \in \mathbf{N}$, there exists $M_n \in \mathbf{R}$ such that $|f_n(x)| \leq M_n$ for all $x \in X$. Suppose further that f_n converges uniformly on X. Then choosing $\epsilon = 1$, there exists $N \in \mathbf{N}$ such that

$$|f_n(x) - f_m(x)| < 1$$

for all $n, m \geq N$, $x \in X$. Taking, in particular, $m = N$ gives us that

$$|f_n(x)| \leq |f_n(x) - f_N(x) + f_N(x)| \leq |f_n(x) - f_N(x)| + |f_N(x)| \leq 1 + M_N$$

for all $x \in X$, $n \geq N$. Therefore, if $M = \max\{M_1, M_2 \ldots, M_{N-1}, M_N + 1\}$, we have

$$|f_n(x)| \leq M$$

for all $x \in X$ and all $n \in \mathbf{N}$. □

Solution to #4 on page 165. The series

$$\sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}$$

diverges when $x = 0$ because the terms are all 1 and do not converge to 0. For each $n \in \mathbf{N}$, the series has an ill-defined term when $x = -1/n^2$, so the series does not converge for these values of x either. On the other hand, if $I \subset \mathbf{R}$ is an interval such that $I \cap \{-1/n^2\}_{n\in\mathbf{N}} = \emptyset$ and $0 \notin \overline{I}$, then then I claim that the series converges uniformly and absolutely on I—in particular, the series converges at every non-zero point in $\mathbf{R} - \{1/n^2\}_{n\in\mathbf{N}}$. To see this is so, observe that since $0 \notin \overline{I}$, there exists $r > 0$ such that $|x| > r$ for all $x \in I$. Thus

$$\left|\frac{1}{1 + n^2 x}\right| \leq \frac{1}{n^2|x| - 1} \leq \frac{1}{rn^2 - 1},$$

for all $x \in I$. Hence, for $n \geq N_1 \geq 1/\sqrt{r - 1/2}$, we have $rn^2 - 1 \geq rn^2/2$ and

$$\left|\frac{1}{1+n^2 x}\right| \leq \frac{2}{r}\frac{1}{n^2}.$$

Let $s_n(x)$ denote the nth partial sum of the above series. Let $\epsilon > 0$. Since $\sum_{n=0}^{\infty} \frac{1}{n^2}$ is convergent, there exists $N_2 \in \mathbf{N}$ such that $m \geq n \geq N_2$ implies that

$$\sum_{k=n}^{m+1} \frac{1}{n^2} < \frac{r\epsilon}{2}.$$

So for $x \in I$, we have for $m \geq n \geq N := \max\{N_1, N_2\}$ that

$$|s_n(x) - s_m(x)| \leq \sum_{k=n+1}^{m} \left|\frac{1}{1+n^2 x}\right| \leq \frac{2}{r} \sum_{k=n}^{m+1} \frac{1}{n^2} < \epsilon.$$

That is, the sequence of partial sums $\{s_n(x)\}_{n\in\mathbf{N}}$ is uniformly Cauchy, and therefore uniformly convergent. This proves my claim. By Theorem 7.12, the series is continous as a function of x on I. Taking the union of all such intervals I tells us that the series defines a continuous function of x on $\mathbf{R} - \{0\} - \{1/n^2\}_{n\in\mathbf{N}}$. □