Supplementary problems (assigned 9/19/03)

1. Rudin's definition of a connected subset of a metric space is a little non-standard. Prove that his definition is equivalent to the following more standard one:

   A subset E of a metric space X is connected if for every pair $U, V \subset X$ of disjoint non-empty open sets whose union contains E, we have either $E \subset U$ or $E \subset V$.

   In other words show that E satisfies Rudin's definition if and only if it satisfies this one.

2. Let $P \subset [0, 1]$ be the middle thirds Cantor set discussed in Rudin and in class.

   (a) Show that P is *totally disconnected*. That is, for every two points $x \neq y$ in P, there are open sets $U, V \subset \mathbf{R}$ such that $x \in U$, $y \in V$ and $P \subset U \cup V$.

   (b) Compute the sum of the lengths of the open intervals in $[0, 1] - P$. Based on your computation, if we were to assign a length to the Cantor set itself, what would it have to be?

   (c) (Extra Credit) Show that the function

   $$(a_j)_{j \in \mathbf{N}} \mapsto \sum_{j=1}^{\infty} \frac{2a_j}{3^j}.$$

   is a bijection from $2^{\mathbf{N}}$ onto P. (Note that in some sense, this gives a description of points P–a point $p \in [0, 1]$ belongs to P if and only if it is given as a ternary decimal expansion whose digits are all zeroes and twos.)

3. Show that the sequence $\{\sin n\}_{n \in \mathbf{N}} \subset \mathbf{R}$ diverges.