Supplementary problems (assigned 11/3/03)

1. Let $f : [a, b] \to \mathbf{R}$ be a bounded function which is continuous at all points $x \in [a, b]$ except for those in a set $S \subset [a, b]$. Suppose that S has the following property: *for every* $\epsilon > 0$*, there is a finite collection* $I_1, \ldots, I_n$ *of disjoint open intervals such that*

   - $S \subset I_1 \cup \ldots I_n$*;*
   - $|I_1| + \ldots |I_n| < \epsilon$*, where* $|I_j|$ *is the length of the interval* I_j*.*

   Show that f is Riemann integrable. (Note that this implies that a function which is continuous except on the Cantor set will be integrable!)

2. Let $f : [0, 1] \to \mathbf{R}$ be the restriction to $[0, 1]$ of the function given in problem 18 on page 100. Show that f is integrable.

   Remarks (not part of the problem): nevertheless, f does not satisfy the hypotheses of the previous problem—you'd need finitely many intervals that covered $\mathbf{Q} \cap [0, 1]$, and it can be shown that the union of these intervals can omit only finitely many points in $[0, 1]$. Hence the sum of the lengths of the intervals couldn't possibly be less than one. On the other hand, you can cover $\mathbf{Q} \cap [0, 1]$ with *countably* many disjoint open intervals the sum of whose lengths is smaller than any given ϵ. This need to allow for countably many intervals is one of the keys to measure theory.

3. (Integrals and series) Suppose that we are given an infinite series $\sum_{k=1}^{\infty} a_k$. Suppose moreover that $a_k = f(k)$ where $f : [0, \infty) \to \mathbf{R}$ is a non-negative, decreasing function.

   - Show that $\int_{n-1}^{m} f(x)\,dx \geq \sum_{k=n}^{m} a_k \geq \int_{n}^{m+1} f(x)\,dx$.
   - Use this to show that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ converges. (Note that the initial values of n and x are irrelevant—i.e. it's not a problem that our series starts with $n = 2$ here.)
   - Note that the first item in this problem can be used to give an upper bound on the difference between the full series and a given partial sum. Use this bound to estimate how many terms of $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ that you'd have to add up to be within .01 of the sum of the full infinite series. Assuming you had a computer handy that was capable of adding up a trillion terms per second, how many years would it take your computer to add up this many terms?
   - Now notice that you can use also use the first item to give a *lower* bound on the difference between the series and a given partial sum. These bounds allow you to get a good handle on the sum of the remaining terms in the series. Use this idea to estimate the value of the full series to within .01 in a more practical fashion (i.e. by Monday 11/10).