Supplementary problems (assigned 11/3/03)

1. (Leftover differentiation problem) Suppose that $f : (a, b) \to \mathbf{R}$ is a convex function that is differentiable at every $x \in (a, b)$. Show that f' is continuous. (Hint: you can, of course, use the results of previous homework problems about convexity; moreover, there is a theorem in the book that makes this problem much easier—for once, it is not the mean value theorem or the chain rule.)

2. The function $1/t$ is continuous on $(0, \infty)$. Therefore the function

$$f(x) = \int_1^x \frac{dt}{t}.$$

is well-defined for all $x \in (0, \infty)$. Prove each of the following about f.

- f is differentiable at every point and strictly increasing.
- $f(xy) = f(x) + f(y)$ for every $x, y \in (0, \infty)$. It helps here to think of y as a constant so that both sides are functions of x only.
- $f(x^t) = tf(x)$ for all $t \in \mathbf{R}, x \in (0, \infty)$. Remember the problem from the first chapter in which x^t was defined for any real t—the idea was to do it first for $t \in \mathbf{Z}$, then for $t \in \mathbf{Q}$, and then, using supremums, for $t \in \mathbf{R}$.
- $f(0, \infty) = \mathbf{R}$. In particular, there is a unique number $d \in (1, \infty)$ such that $f(d) = 1$.
- f is an invertible function and that $f^{-1}(y) = d^y$ for all $y \in \mathbf{R}$.