Math 366, Final Test

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May 8, 1997
Instructor: Bei Hu

Name:______________________________

1. (30 points).

(a). State PRECISELY the definition of an k-form.

(b). State PRECISELY the Stone-Weierstrass theorem in a rectangle in R^2.

(c). State PRECISELY the Stocks theorem.

(d). State PRECISELY the theorem of partition of unity.

(e). State PRECISELY a necessary and sufficient condition for the existence of a Riemann-Stieltjes integral.

(f). State PRECISELY the Ascoli Arzelá theorem.

2. (12 points). In R^3, we use (x, y, z) variables. Let $= y dx \wedge dz$. Let Φ be a 2-surface $x = u_1, y = u_2, z = 1 - u_1 - u_2$, $u_1 \geq 0$, $u_2 \geq 0$, $u_1 + u_2 \leq 1$. Find

$$\int_\Phi .$$

3. (12 points). Is it true that every uniformly convergent sequence of bounded functions is uniformly bounded? Prove or give a counter example.

4. (12 points)

Suppose that f is Riemann integrable on $[0, A]$ for any $A > 0$ and $f(x) \to 1$ as $x \to +\infty$. Prove that

$$\lim_{t \to 0+} t \int_0^\infty e^{-tx} f(x) dx = 1.$$

5. (12 points)

Consider

$$f_1(x_1, x_2, y_1, y_2, y_3) = e^{x_1} + x_2 y_2 - 9 y_1 y_3 = 0$$
$$f_2(x_1, x_2, y_1, y_2, y_3) = 5 x_2 g(x_1) + g(x_1 + 2) + 2 y_2 + 8 y_3 = 0,$$

where $g(x_1)$ is continuously differentiable. We want to solve (x_1, x_2) in terms of (y_1, y_2, y_3) near $(0, 0, 0, 0, 0)$.

Find the condition(s) on $g(x_1)$ so that the implicit function theorem can be used near $(0, 0, 0, 0, 0)$.

6. (12 points). Suppose that $= -ydx + xdx$. Suppose that $\emptyset$ is a simply connected domain with smooth boundary. What can you conclude from the Stokes theorem?

Explain your result.

7. (12 points). Consider the family of continuous functions $f_n(x,y) = \frac{x^2+y^2+5xy}{x^2+y^2+(1/n)^2}$ on the disk $B_1 = \{(x,y) | x^2+y^2 \le 1\}$.

It is clear that $f_n(x,y)$ converges to some $f(x,y)$ pointwise on B_1.

(a) Show that $f(x,y)$ is not continuous.

(b) Is the convergence $f_n(x,y) \to f(x,y)$ uniform on B_1? Justify your result.

8. (10 points). Let $f(x) = x + e^{-x/2}$ and $X = [0, \infty)$. We claim that $f(x)$ has a fixed point in X, by the contraction mapping principle.

Proof. It is clear that for $x \in X$, $f'(x) = 1 - \frac{1}{2}e^{-x}$, and therefore $0 < f'(x) < 1$. It follows that, by Mean value theorem

$$|f(x) - f(y)| = |f'(\xi)| < 1x, y \in X.$$

Therefore the contraction mapping principle will give a unique fixed point for $f(x)$.

Is this proof correct? Why or why not.

9. (10 points). Suppose that $f(x) \geq 0$ for $x \geq 0$ and is monotonically decreasing. Prove that $\int_0^\infty f(x)dx$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges.

10. (8 points). Suppose that $f(x)$ is continuous and

$$\int_E f(x)dx = 0$$

for any open set E in R^n. Prove that $f(x) \equiv 0$.