Math 366, Test # 1   / 100

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Name:______________________________

1. (30 points). True or False. (No proof or counter example needed).

(a). Let α be monotonically increasing and continuous on $[a, b]$. Suppose f is bounded and monotone, then f is Riemann-Stieltjes integrable with respect to α over $[a, b]$.

(b). If $\gamma(t)$ is continuously differentiable on $t \in [a, b]$, then γ is rectifiable.

(c). The limit function of a sequence of continuous functions is always continuous.

(d). If $\{f_n(x)\}$ and $\{g_n(x)\}$ converge uniformly on a set E, then their product $\{f_n g_n\}$ also converges uniformly on E.

(e). Suppose $f_n(x)$ are continuous functions, $f_n(x)$ converges to $f(x)$ pointwise on $[a, b]$ and x_n converges to $c \in (a, b)$. Then

$$\lim_{n\to\infty} f_n(x_n) = f(c).$$

(f). If $\{f_n(x)\}$ converges to $f(x)$ pointwise on $[a, b]$ and $f_n \in \mathcal{R}(\alpha)$, then $f \in \mathcal{R}(\alpha)$.

2. (20 points).

(a). State PRECISELY the definition of a Riemann-Stieltjes integral, using Partition.

(b). State PRECISELY the definition of a Rectifiable curve.

(c). State PRECISELY the M-test for uniform convergence of a series.

(d). State PRECISELY the definition of equi-continuity of a family of functions.

3. (10 points). Suppose that $p > 0$, $q > 0$ and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

For $u \geq 0$, $v \geq 0$, prove that

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

4. (10 points). Let

$$f_n(x) = \frac{x}{1+n^2x^2}.$$

It is clear that $f_n(x)$ converges to 0 pointwise.

Does $\{f_n\}$ converge uniformly over the interval $[0,1]$? Prove your claim.

5. (10 points). Suppose that $\int_a^{a+1} e^{u(x)}dx = 1$, and that $u(a) = u(a+1) = 0$. Evaluate the integral

$$\int_a^{a+1} xe^{u(x)}u'(x)dx$$

6. (10 points). Let $f(x) = 1$ for $0 < x \leq 1$, and $f(0) = 0$.

Can you find a sequence of continuous functions $f_n(x)$ defined on $[0, 1]$ such that,

(a) $f_n(x)$ converges to $f(x)$ pointwise on $[0, 1]$. Prove your answer.

(b) $f_n(x)$ converges to $f(x)$ uniformly on $[0, 1]$. Prove your answer.

7. (10 points). Suppose that f_n and f are continuous functions. f_n converges to f pointwise on $[0, 1]$. Assume that $f_n(x) \leq f_{n+1}(x)$ $(n = 1, 2, 3 \cdots)$. Prove that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx.$$